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SLICING SITES AND SEMIREPLETE FACTORIZATION SYSTEMS

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ABSTRACT. A factorization system $(\mathcal{E}, \mathcal{M})$ on a category $\mathscr{A}$ gives rise to the covariant category-valued pseudofunctor P of $\mathscr{A}$ sending each object to its slice category over $\mathcal{M}$. This article characterizes the P so obtained as follows: their object images have terminal objects, and they admit bicategorically cartesian liftings, up to equivalence, of slice-category projections. It is clear that, and how, $(\mathcal{E}, \mathcal{M})$ can be recovered from such a P. The correspondence thus described is actually the second of three similar ones between certain $(\mathcal{E}, \mathcal{M})$ and certain P that the article presents. In the first one the characterization of the P has all ultimately bicategorical ingredients replaced with their categorical analogues. A category $\mathscr{A}$ with such a P is precisely what the author has called a "slicing site". In the article's terms the associated $(\mathcal{E}, \mathcal{M})$ are again factorization systems — but the concept conveyed extends the standard one by not obliging isomorphisms to belong to either factor class —, namely those that are "right semireplete" (isomorphisms do belong to $\mathcal{M}$) and "left semistrict" (morphisms in $\mathcal{M}$ are monic relative to $\mathcal{E}$). The third correspondence subsumes the other two; here the $(\mathcal{E}, \mathcal{M})$ are all right-semireplete factorization systems.

Introduction

Every definition of higher-dimensional categories seriously considered to this day has its underlying concept of higher-dimensional "graphs", featuring cells and their boundary relations. Traditionally this is simply globular sets, but more complicated devices have also shown merits, such as (semi-)simplicial sets or (semi-)cubical sets. The higher-dimensional graphs always form a presheaf category, the site being the category of relevant cell "shapes" (globes, simplices, cubes). Usually they are so defined, as in the cases just mentioned, while in the cases of opetopic sets ([1]) and of the variant concept of multitopic sets ([6]) the statement requires proof.

This necessity also arises for a certain concept related to the last two mentioned (actually equivalent to the latter), namely that of dendrotopic sets, which I introduced in [8]. Here the cell shapes can be described conveniently as special instances; a trait inherited from the auxillary concept of *polytopic sets*, where the cell shapes are called *polytopes*. One may tend to think of polytopic sets as primordeal higher-dimensional graphs: in the cited paper I indicated how the higher-dimensional graphs according to each of the above concepts can be viewed as polytopic sets with appropriate extra structure, foremost orientations. Yet polytopic sets without (or with only a fragment of this) extra structure fail

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to form a presheaf category, the obstacle being non-trivial morphisms between polytopes of the same dimension.

I gave a precise account of this matter in the article [9]. The present paper I have conceived as the first instalment in a series of two, [10] to be the second, delivering substantial enhancements of those earlier results.

That account concerned far more generally a *category with slicing*: a category $\mathcal{E}$ together with a functor ∂ to **Cat** satisfying axioms stating, roughly, that the slice-category construction can be lifted universally along ∂, so that we obtain a satisfactory slice-*object* construction for $\mathcal{E}$. The "ideal" example is a presheaf category together with its category-of-elements functor, the slice objects being the represented presheaves given by the Yoneda lemma.

The main theorem told of a canonical embedding of $\mathcal{E}$ in a presheaf category $\mathbf{Set}^{\check{\mathcal{E}}^{\mathrm{op}}}$ and characterized the presheaves in its image. The site $\check{\mathcal{E}}$ is defined as follows. Call an object of $\mathcal{E}$ represented if it comes equipped with a distinguished terminal object, the representation, of its ∂-image. The objects of $\check{\mathcal{E}}$ are the represented objects of $\mathcal{E}$; their morphisms in $\check{\mathcal{E}}$ are all their morphisms in $\mathcal{E}$. Of course this definition is odd by categorists' standards, as the objects carry structure that is ignored by the morphisms. Said embedding is induced (assuming $\mathcal{E}$ is locally small) by the functor $\check{\mathcal{E}} \longrightarrow \mathcal{E}$ forgetting representations. Said characterization is given in terms of two kinds of special morphisms in $\check{\mathcal{E}}$: the *discrete fibrations*, which are those ∂-cartesian morphisms whose ∂-images are ordinary discrete fibrations, and the *representation-preserving* ones, which are those morphisms whose ∂-images do as the name suggests ("on the nose"). The latter are the ones we should regard as the "true" morphisms of the objects at hand. Yet in the case of a presheaf category and its category-of-elements functor the category of represented objects and representation-preserving morphisms is essentially discrete, a property that allows conversely to conclude from the theorem that the canonical embedding is an equivalence.

So the theorem allows for $\mathcal{E}$ as a mere category to be reconstructed from $\check{\mathcal{E}}$ as a category with two distinguished morphism classes, say $\check{\mathcal{E}}_{\text{di.fi.}}$ and $\check{\mathcal{E}}_{\text{re.pr.}}$. The question arises whether ∂ as well can be reconstructed. The affirmative answer given in the article was evasive in that it referred to additional data much more closely related to ∂, namely the corresponding functor $\check{\partial}$ to the accordingly defined category $\check{\mathbf{Cat}}$([1]). (One can arrange the pertinent data to a pullback square with $\check{\mathcal{E}}$ as the vertex.) $\check{\partial}$ admits cartesian liftings, up to representation-preserving isomorphism, of slice-category projections. This property, inherited from the principal axiom on a category with slicing, was taken as the single axiom on a *slicing site*.

Clearly $\check{\partial}$ suffices to determine $\check{\mathcal{E}}_{\text{di.fi.}}$ and $\check{\mathcal{E}}_{\text{re.pr.}}$. The enhancement delivered in the present work is that the converse is also true: from $\check{\mathcal{E}}_{\text{di.fi.}}$ and $\check{\mathcal{E}}_{\text{re.pr.}}$ (even from $\check{\mathcal{E}}_{\text{re.pr.}}$ alone) we can reconstruct $\check{\partial}$. In conjunction with the prior result this yields a proper affirmative answer to the question as stated. The eager reader will find no big difficulties in replacing the objects of $\check{\mathcal{E}}$ with the relevant presheaves on $\check{\mathcal{E}}$ to obtain a direct reconstruction of ∂

[1] People with some knowledge of Czech will pronounce this symbol like the English word 'chat'.

that can serve as an adequate supplement to the theorem. This idea will be pursued in the mentioned second instalment.

As a prelude to the reconstruction of $\check{\partial}$ we have to find requisite properties of $\check{\mathcal{E}}_{\text{di.fi.}}$ and $\check{\mathcal{E}}_{\text{re.pr.}}$. The proof of the theorem decisively used what I called the *slice-object factorization*, which in particular exhibits an arbitrary morphism in a slicing site as the composite of first a representation-preserving one and then a discrete fibration. This would suggest that $(\check{\mathcal{E}}_{\text{re.pr.}}, \check{\mathcal{E}}_{\text{di.fi.}})$ is a factorization system — if it were not for the evident failure, in general, of an isomorphism to be representation preserving; a reminder of the oddity of the definition of $\check{\mathcal{E}}$.

Upon closer inspection this failure turns out to be the only one. Emboldened by the example I widen the scope of the term 'factorization system' by dropping the condition that isomorphisms belong to either factor class (thus retaining self-duality). If they nevertheless do, I call the factorization system *replete*. If they at least belong to the right-factor class([2]), as in the example at hand, I call the factorization system *right semireplete*.

In the construction itself each object of $\check{\mathcal{E}}$ is assigned a skeleton with respect to isomorphisms from $\check{\mathcal{E}}_{\text{re.pr.}}$ of the slice category over $\check{\mathcal{E}}_{\text{di.fi.}}$. But for an arbitrary right-semireplete factorization system this merely yields a pseudofunctor. To obtain an actual functor we rely on yet another property of $\check{\mathcal{E}}_{\text{di.fi.}}$ and $\check{\mathcal{E}}_{\text{re.pr.}}$: the right factors are monic relative to the left ones; that is, an individual factorization is determined by its right half. I call such a factorization system *left semistrict*.

This, then, is all (apart from the issue of size): for every left-semistrict, right-semireplete factorization system the construction yields a slicing-site structure inducing it. So in effect we obtain a conceptually simpler definition of 'slicing site'.

This summarizes the content of section 1, which makes up most of this work. Section 2 describes the 2-category of slicing sites in terms of the associated factorization systems and goes on to consider arbitrary functors and natural transformations as the higher cells of a convenient larger realm of discourse. Section 3 tells how to maintain the foregoing results while dropping the unusual condition of left semistrictness. The second instalment will present related results on categories with slicing.

Note on terminology. We shall have to deal with some 2- (most actually bi-)categorical concepts, albeit only very basic ones. When it comes to the 2-dimensional generalizations of a 1-dimensional concept, distinguished by how well the requisite laws are obeyed — without qualification, up to invertible cells, up to arbitrary cells the conventional way or up to arbitrary cells the opposite way —, I am going to employ the terms '*strict(-ly)*', '*weak(-ly)*', '*lax(-ly)*' and '*op-lax(-ly)*', respectively, the first one usually being a default. So I say 'weak functor' rather than 'pseudofunctor' (in spite of the above) or 'homomorphism', and I say 'weakly natural transformation' rather than 'pseudonatural transformation' or 'strong transformation'. Thus there always is an adjective available to express any of these qualities with regard to mediating cells already present (as in 'this

[2]The terms 'left' and 'right' will refer to the graphical representation using horizontal arrows that, as usual, run in the reading direction common to all Western languages.

op-lax functor is weak' or 'this laxly natural transformation is weakly natural').

1. Objects

Factorization systems

Let us start with a review of a well-known concept. A *factorization system* on a category $\mathscr{A}$ consists of two classes of morphisms in $\mathscr{A}$, say $\mathcal{L}$ and $\mathcal{R}$, obeying certain requirements. We call the elements of $\mathcal{L}$ *left morphisms* and represent them graphically by the symbol '$\twoheadrightarrow$', and we call the elements of $\mathcal{R}$ *right morphisms* and represent them graphically by the symbol '$\rightarrowtail$'. Some basic properties are given as (F1–6) below. That list has been compiled with a certain purpose in mind. It can serve as an axiomatization; as such, however, it is marred by several logical dependencies. All of this will soon be made clear.

$(\mathrm{F1})_{\mathrm{l}}$ $\mathcal{L}$ contains all identities of $\mathscr{A}$.

$(\mathrm{F1})_{\mathrm{r}}$ $\mathcal{R}$ contains all identities of $\mathscr{A}$.

$(\mathrm{F2})_{\mathrm{l}}$ $\mathcal{L}$ is closed in $\mathscr{A}$ with respect to binary composition.

$(\mathrm{F2})_{\mathrm{r}}$ $\mathcal{R}$ is closed in $\mathscr{A}$ with respect to binary composition.

$(\mathrm{F3})_{\mathrm{l}}$ For every composable two morphisms l and g in $\mathscr{A}$, if $l \in \mathcal{L}$ and $g{\cdot}l \in \mathcal{L}$, then also $g \in \mathcal{L}$.

$(\mathrm{F3})_{\mathrm{r}}$ For every composable two morphisms g and r in $\mathscr{A}$, if $r \in \mathcal{R}$ and $r{\cdot}g \in \mathcal{R}$, then also $g \in \mathcal{R}$.

(F4) Every morphism in $\mathscr{A}$ is a composite $r{\cdot}l$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$.

(F5) For every $l \in \mathcal{L}$ and every $r \in \mathcal{R}$, l is *left orthogonal* to r, or, equivalently, r is *right orthogonal* to l. That is, for each commutative square

$$\begin{array}{ccc} D & \xrightarrow{h} & B \\ {\scriptstyle l}\big\downarrow & \overset{g}{\nearrow} & \big\downarrow{\scriptstyle r} \\ C & \xrightarrow[f]{} & A, \end{array} \tag{1}$$

there is a unique morphism g as shown, rendering the two triangles commutative.

$(\mathrm{F6})_{\mathrm{l}}$ $\mathcal{L}$ contains all invertible morphisms of $\mathscr{A}$.

$(\mathrm{F6})_{\mathrm{r}}$ $\mathcal{R}$ contains all invertible morphisms of $\mathscr{A}$.

This description is self-dual in the obvious sense; in fact for each occurrence of $(\mathrm{F}i)$ (without subscript) the statement so denoted is self-dual, while for each occurrence of $(\mathrm{F}j)_{\mathrm{l}}$ and $(\mathrm{F}j)_{\mathrm{r}}$ the two statements so denoted are dual to each other (and so their conjunction, to be denoted by just $(\mathrm{F}j)$, is self-dual). Statements $(\mathrm{F1,2})_{\mathrm{l}}$ and statements $(\mathrm{F1,2})_{\mathrm{r}}$ can be subsumed by saying that $\mathcal{L}$ and $\mathcal{R}$, respectively, are all-object subcategories of $\mathscr{A}$.

For axiomatization purposes the list can be pruned down as follows. Conditions $(\mathrm{F1})_{\mathrm{l}}$ and $(\mathrm{F1})_{\mathrm{r}}$ can obviously be omitted. (They are superseded by $(\mathrm{F6})_{\mathrm{l}}$ and $(\mathrm{F6})_{\mathrm{r}}$, respectively.) Conditions $(\mathrm{F3})_{\mathrm{l}}$ and $(\mathrm{F3})_{\mathrm{r}}$ can also be omitted, but less obviously so. (Consider the latter: it is implied by $(\mathrm{F2})_{\mathrm{r}}$ and (F4) along with its weaker variant $(\mathrm{F3})'_{\mathrm{r}}$ obtained by adding '$g \in \mathcal{L}$' to the premiss; $(\mathrm{F3})'_{\mathrm{r}}$ is in turn implied by (F5) and $(\mathrm{F6})_{\mathrm{r}}$.) Moreover, condition (F5) can be replaced with its weaker variant $(\mathrm{F5})'$ obtained from its spelt-out form by adding '$h \in \mathcal{L}$ and $f \in \mathcal{R}$' to the premiss. (It is implied by (F2), (F4) and $(\mathrm{F5})'$.)

The further discussion is facilitated by some notation. ("Calligraphic" letters will stand for arbitrary morphism classes.) We denote by $\mathscr{A}^0$ the class of all identities in $\mathscr{A}$, by $\mathcal{F}{\cdot}\mathcal{H}$ the class of all composites $f{\cdot}h$ with $f \in \mathcal{F}$ and $h \in \mathcal{H}$, by $\mathcal{F}^{\perp}$ the class of morphisms left orthogonal to all of $\mathcal{F}$, by ${}^{\perp}\mathcal{H}$ the class of morphisms right orthogonal to all of $\mathcal{H}$, and by $\mathscr{A}^{\times}$ the class of all invertible morphisms in $\mathscr{A}$. Now we can write most of the above conditions in symbolic form as inclusions: $(\mathrm{F1})_{\mathrm{l}}$ is $\mathscr{A}^0 \subseteq \mathcal{L}$, $(\mathrm{F1})_{\mathrm{r}}$ is $\mathscr{A}^0 \subseteq \mathcal{R}$ (and (F1) is $\mathscr{A}^0 \subseteq \mathcal{L} \cap \mathcal{R}$); $(\mathrm{F2})_{\mathrm{l}}$ is $\mathcal{L}{\cdot}\mathcal{L} \subseteq \mathcal{L}$, $(\mathrm{F2})_{\mathrm{r}}$ is $\mathcal{R}{\cdot}\mathcal{R} \subseteq \mathcal{R}$; (F4) is $\mathscr{A} \subseteq \mathcal{R}{\cdot}\mathcal{L}$; (F5) is $\mathcal{L} \subseteq \mathcal{R}^{\perp}$, or, equivalently, $\mathcal{R} \subseteq {}^{\perp}\mathcal{L}$; $(\mathrm{F6})_{\mathrm{l}}$ is $\mathscr{A}^{\times} \subseteq \mathcal{L}$, $(\mathrm{F6})_{\mathrm{r}}$ is $\mathscr{A}^{\times} \subseteq \mathcal{R}$ (and (F6) is $\mathscr{A}^{\times} \subseteq \mathcal{L} \cap \mathcal{R}$).

The converse of (F6) holds true (it follows from a much weaker variant of (F5)), so that in fact $\mathscr{A}^{\times} = \mathcal{L} \cap \mathcal{R}$. Also, the two converses of (F5) hold true (as for ${}^{\perp}\mathcal{L} \subseteq \mathcal{R}$, the inclusion ${}^{\perp}\mathcal{L} \subseteq \mathcal{R}{\cdot}\mathscr{A}^{\times}$ follows from (F4) and (F5), and $\mathcal{R}{\cdot}\mathscr{A}^{\times} \subseteq \mathcal{R}$ trivially follows from $(\mathrm{F2})_{\mathrm{r}}$ and $(\mathrm{F6})_{\mathrm{r}}$), so that in fact

$$ {}^{\perp}\mathcal{L} = \mathcal{R} \quad \text{and} \quad \mathcal{R}^{\perp} = \mathcal{L}. \tag{2} $$

Conversely, any class of the form $\mathcal{F}^{\perp}$ (in place of $\mathcal{L}$) satisfies all of the above "left conditions", and any class of the form ${}^{\perp}\mathcal{H}$ (in place of $\mathcal{R}$) satisfies all of the above "right conditions". Thus we see that the entire list is equivalent to just (F4) and (2), which are well known to form the snappiest axiomatization of 'factorization system'.

Condition $(\mathrm{F5})'$, its premiss being symmetric in (l, f) and (h, r), implies that the morphism g in its conclusion is invertible; the proof uses the most basic among nontrivial category-theoretic arguments. We call the condition and this consequence together the *symmetric variant* of (F5). We may render the former by adapting the common-parlance phrase 'unique up to isomorphism': an $(\mathcal{L}, \mathcal{R})$-factorization is *unique up to a unique morphism*. To convey the latter at the same time we can employ a parenthesis: 'unique up to a unique *(iso-)*morphism'. This exemplifies how we are going to deal with analogous situations below.

Other preliminaries

We continue with a concise account of those concepts fundamental to this work that appear in [9].

Let $\mathscr{A}$ and $\mathscr{K}$ be two categories, and let $P : \mathscr{A} \longrightarrow \mathscr{K}$ be a functor. This situation is described by saying that $(\mathscr{A}, P)$ is a category over $\mathscr{K}$. In most of the following terminology P is understood. Let $f : B \longrightarrow A$ be a morphism in $\mathscr{A}$. It is said to lie *above* a morphism $f' : B' \longrightarrow A'$ in $\mathscr{K}$ if $Pf = f'$. It is called *cartesian* if for every $g : C \longrightarrow A$

in $\mathscr{A}$ and $h' : PC \longrightarrow PB$ in $\mathscr{K}$ with $Pf{\cdot}h' = Pg$ there is a unique $h : C \longrightarrow B$ above h' with $f{\cdot}h = g$. Now let A be an object of $\mathscr{A}$ and $f' : B' \longrightarrow PA$ be a morphism in $\mathscr{K}$. A cartesian (*straight*) *lifting* of f' *for* A consists of an object B of $\mathscr{A}$ and a cartesian morphism $f : B \longrightarrow A$ above f'; if one exists for all A and f' then P is a Grothendieck fibration. A cartesian *up-to-isomorphism* lifting of f' for A consists of an object B of $\mathscr{A}$, a cartesian morphism $f : B \longrightarrow A$, and, here called its *skewness*, an isomorphism $\beta' : PB \xrightarrow{\simeq} B'$ in $\mathscr{K}$ with $Pf = f'{\cdot}\beta'$; if one exists for all A and f' then P is a Street fibration. The cartesianness condition is equivalent to a similar one involving the original f': for every $g : C \longrightarrow A$ in $\mathscr{A}$ and $h' : PC \longrightarrow B'$ in $\mathscr{K}$ with $f'{\cdot}h' = Pg$ there is a unique $h : C \longrightarrow B$ in $\mathscr{A}$ with $\beta'{\cdot}Ph = Pf$ and $f{\cdot}h = g$. (This exhibits a cartesian up-to-isomorphism lifting for A as being the same as an invertible universal morphism from the induced functor $P{\downarrow}A : \mathscr{A}{\downarrow}A \longrightarrow \mathscr{K}{\downarrow}PA$.) Such a lifting is unique up to a unique (iso-)morphism; that is, for any two cartesian up-to-isomorphism liftings (B_0, f_0, β'_0) and (B_1, f_1, β'_1) of f' for A there is a unique (iso-)morphism $\beta : B_0 \longrightarrow B_1$with $\beta'_1{\cdot}P\beta = \beta'_0$ and $f_1{\cdot}\beta = f_0$. We can look at cartesian liftings up to any class $\mathcal{X}'$ of isomorphisms in $\mathscr{K}$ (that is, we require the skewnesses to be in $\mathcal{X}'$). Preferably $\mathcal{X}'$ is an all-object subgroupoid of $\mathscr{K}$; then the isomorphism mediating between two such liftings lies above one in $\mathcal{X}'$. Taking $\mathcal{X}'$ to consist of just the identities yields the concept of a cartesian (strict) lifting; taking $\mathcal{X}'$ to consist of all isomorphisms yields the concept of a cartesian up-to-isomorphism lifting.

We are interested in categories over **Cat**, the category of (small) categories and functors. It will be advantageous to make certain typographical distinctions between categories occurring in their own right and categories occurring as objects of **Cat**. Primarily, while the former appear in script type (apart from the use of object-name abbreviations such as '**Cat**'), the latter will appear in ordinary italic type, as do objects of the former in general; objects of the latter will appear in serifless type. In what follows this convention covers for overt **Cat**-membership statements.

Here is a quick run-down of the pertinent features of slicing in **Cat**. Let A be a category. Each object A of A gives rise to the *slice category* $A{\downarrow}\mathsf{A}$ of objects over A. This has objects (X, x) with $\mathsf{X} \in A$ and $\mathsf{x} : \mathsf{X} \longrightarrow \mathsf{A}$, and morphisms $\mathsf{i} : (\mathsf{X}_0, \mathsf{x}_0) \longrightarrow (\mathsf{X}_1, \mathsf{x}_1)$ with $\mathsf{i} : \mathsf{X}_0 \longrightarrow \mathsf{X}_1$ such that $\mathsf{x}_1 \circ \mathsf{i} = \mathsf{x}_0$. We denote the associated projection $A{\downarrow}\mathsf{A} \longrightarrow A$ by δ^A_{A}. The functor δ^A_{A} is invertible precisely if A is terminal. Each morphism $\mathsf{u} : \mathsf{A}' \longrightarrow \mathsf{A}$ in A induces a functor $A{\downarrow}\mathsf{A}' \longrightarrow A{\downarrow}\mathsf{A}$ (namely, $(\mathsf{X}', \mathsf{x}') \longmapsto (\mathsf{X}', \mathsf{u} \circ \mathsf{x}')$ and $\mathsf{i}' \longmapsto \mathsf{i}'$), which we denote by $A{\downarrow}\mathsf{u}$. We have $A{\downarrow}1_{\mathsf{A}} = 1_{A{\downarrow}\mathsf{A}}$ and $A{\downarrow}(\mathsf{u} \circ \mathsf{u}') = (A{\downarrow}\mathsf{u}){\cdot}(A{\downarrow}\mathsf{u}')$ (that is, $A{\downarrow}()$ is an A-shaped diagram in **Cat**), and further $\delta^A_{\mathsf{A}}{\cdot}(A{\downarrow}\mathsf{u}) = \delta^A_{\mathsf{A}'}$ (that is, $\delta^A_{()}$ is a cocone $A{\downarrow}() \longrightarrow A$). Now let $f : B \longrightarrow A$ be a functor. For each object B of B it induces a functor $B{\downarrow}\mathsf{B} \longrightarrow A{\downarrow}f(\mathsf{B})$ (namely, $(\mathsf{Y}, \mathsf{y}) \longmapsto (f(\mathsf{Y}), f(\mathsf{y}))$ and $\mathsf{j} \longmapsto f(\mathsf{j})$), which we denote by $f{\downarrow}\mathsf{B}$. We have $(A{\downarrow}f(\mathsf{v})){\cdot}(f{\downarrow}\mathsf{B}') = (f{\downarrow}\mathsf{B}){\cdot}(B{\downarrow}\mathsf{v})$ (that is, $f{\downarrow}()$ is a morphism $B{\downarrow}() \longrightarrow A{\downarrow}f(())$ of B-shaped diagrams), and further $\delta^A_{f(\mathsf{B})}{\cdot}(f{\downarrow}\mathsf{B}) = f{\cdot}\delta^B_{\mathsf{B}}$ (that is, $f{\downarrow}()$ and f together are a morphism of cocones). Moreover, for each category A and each object $\mathsf{A} \in A$ we have $1_A{\downarrow}\mathsf{A} = 1_{A{\downarrow}\mathsf{A}}$, and for each composable pair of functors $C \xrightarrow{h} B \xrightarrow{f} A$ and each object $\mathsf{C} \in C$ we have $f{\cdot}h{\downarrow}\mathsf{C} = (f{\downarrow}h(\mathsf{C})){\cdot}(h{\downarrow}\mathsf{C})$. A functor f is an (ordinary) *discrete fibration* precisely if all the $f{\downarrow}\mathsf{B}$ are invertible. All the functors δ^A_{A} are discrete fibrations.

We are even more interested in categories over $\check{\mathbf{Cat}}$, the category made up as follows. An object A is a (small) *represented* category, that is, a category together with a distinguished terminal object, called its *representation* (abbreviation: *repr.*) and denoted by T_A. A morphism $f : B \longrightarrow A$ is simply a functor; the unique morphism $f(\mathsf{T}_B) \longrightarrow \mathsf{T}_A$ will be denoted by $!_f$. If $!_f$ is an identity, we call f *(strictly) repr. preserving*. If $!_f$ is merely invertible, that is, if f preserves terminality (as an ordinary functor), we call f *weakly repr. preserving*. A strict or weak inverse of a strictly or weakly repr.-preserving functor (that in addition is an isomorphism or an equivalence) is also strictly or weakly repr. preserving, respectively.

What has been said about slicing with respect to $\mathbf{Cat}$ can be said as well with respect to $\check{\mathbf{Cat}}$, with the following addenda. We view $A{\downarrow}\mathsf{A}$ as being represented by $(\mathsf{A}, 1_\mathsf{A})$. The functor δ^A_A is weakly repr. preserving precisely if A is terminal in A, and it is (strictly) repr. preserving precisely if A is the repr. of A. All the functors $f{\downarrow}\mathsf{B}$ are repr. preserving.

We call a morphism in a category over $\check{\mathbf{Cat}}$ strictly or weakly repr. preserving if it lies above a strictly or weakly repr.-preserving functor, respectively. We call a morphism in a category over $\mathbf{Cat}$ a discrete fibration if it is cartesian above an ordinary discrete fibration. Owing to the projection $\check{\mathbf{Cat}} \longrightarrow \mathbf{Cat}$ (forgetting repr.s), every category over $\check{\mathbf{Cat}}$ can be regarded as a category over $\mathbf{Cat}$.

Let $(\mathscr{A}, P)$ be a category over $\check{\mathbf{Cat}}$. Let A be an object of $\mathscr{A}$, and let X be an object of PA. By a *slice object* of objects over X we mean a cartesian up-to-repr.-preserving-isomorphism lifting of δ^{PA}_X for A. Explicitly, this consists of an object $A{\downarrow}\mathsf{X}$ of $\mathscr{A}$, a morphism $\delta^A_\mathsf{X} : A{\downarrow}\mathsf{X} \longrightarrow A$, and a repr.-preserving invertible functor $\varepsilon^A_\mathsf{X} : P(A{\downarrow}\mathsf{X}) \xrightarrow{\simeq} PA{\downarrow}\mathsf{X}$ with $P\delta^A_\mathsf{X} = \delta^{PA}_\mathsf{X}{\cdot}\varepsilon^A_\mathsf{X}$, such that for any object B of $\mathscr{A}$, any morphism $f : B \longrightarrow A$, and any functor $g' : PB \longrightarrow PA{\downarrow}\mathsf{X}$ with $\delta^{PA}_\mathsf{X}{\cdot}g' = Pf$, there is a unique morphism $g : B \longrightarrow A{\downarrow}\mathsf{X}$ with $\delta^A_\mathsf{X}{\cdot}g = f$ and $\varepsilon^A_\mathsf{X}{\cdot}Pg = g'$. We may also more vividly call $A{\downarrow}\mathsf{X}$ on its own a slice object and δ^A_X the associated projection; to ε^A_X we continue to refer as the skewness. The notation introduced here, pretending dependence on just (A and) X, has to be used with caution: while a slice object of objects over X is unique up to a unique repr.-preserving (iso-)morphism, we shall encounter situations in which two of them can differ in a way relevant to us. There always is a slice object of objects over the repr. T_{PA}, namely $(A, 1_A, (\delta^{PA}_{\mathsf{T}_{PA}})^{-1})$.

A *slicing site* is a category $\mathscr{A}$ together with a functor $P : \mathscr{A} \longrightarrow \check{\mathbf{Cat}}$ that admits cartesian up-to-repr.-preserving-isomorphism liftings of all slice-category projections; in other words, a category over $\check{\mathbf{Cat}}$ in which all slice objects exist. If a particular slice object $(A{\downarrow}\mathsf{X}, \delta^A_\mathsf{X}, \varepsilon^A_\mathsf{X})$ has been singled out for every combination of $A \in \mathscr{A}$ and $\mathsf{X} \in PA$ (as is possible using the axiom of choice), we speak of a *global choice* of slice objects. If this has been done such that always $(A{\downarrow}\mathsf{T}_{PA}, \delta^A_{\mathsf{T}_{PA}}, \varepsilon^A_{\mathsf{T}_{PA}}) = (A, 1_A, (\delta^{PA}_{\mathsf{T}_{PA}})^{-1})$, we call the choice *normal*.

Let $(\mathscr{A}, P)$ be a slicing site. Given an object A of $\mathscr{A}$ and a morphism $\mathsf{u} : \mathsf{X}' \longrightarrow \mathsf{X}$ in PA, we define a morphism $A{\downarrow}\mathsf{u} : A{\downarrow}\mathsf{X}' \longrightarrow A{\downarrow}\mathsf{X}$ by demanding that $\delta^A_\mathsf{X}{\cdot}(A{\downarrow}\mathsf{u}) = \delta^A_{\mathsf{X}'}$ and $\varepsilon^A_\mathsf{X}{\cdot}P(A{\downarrow}\mathsf{u}) = (PA{\downarrow}\mathsf{u}){\cdot}\varepsilon^A_{\mathsf{X}'}$. Given a morphism $f : B \longrightarrow A$ in $\mathscr{A}$ and an object Y of PB, we define a morphism $f{\downarrow}\mathsf{Y} : B{\downarrow}\mathsf{Y} \longrightarrow A{\downarrow}Pf(\mathsf{Y})$ by demanding that $\delta^A_{Pf(\mathsf{Y})}{\cdot}(f{\downarrow}\mathsf{Y}) = f{\cdot}\delta^B_{\mathsf{Y}}$ and $\varepsilon^A_{Pf(\mathsf{Y})}{\cdot}P(f{\downarrow}\mathsf{Y}) = (Pf{\downarrow}\mathsf{Y}){\cdot}\varepsilon^B_{\mathsf{Y}}$. All the stated facts on slicing with respect to $\check{\mathbf{C}}\mathbf{at}$ have obvious analogues for slicing with respect to $(\mathscr{A}, P)$.

The one example of a slicing site to keep in mind in order to understand the present work is $\check{\mathbf{C}}\mathbf{at}$, viewed as a category over itself by means of the identity functor.

The slice-object factorization

Any functor $f : B \longrightarrow A$ of represented categories can be factored as

$$B \xrightarrow{f_{\mathsf{T}}} A{\downarrow}\mathsf{X} \xrightarrow{\delta^A_{\mathsf{X}}} A, \tag{3}$$

where $\mathsf{X} = f(\mathsf{T}_B)$, and f_{T} is the functor making the assignments $\mathsf{Y} \longmapsto (f(\mathsf{Y}), f(!_{\mathsf{Y}}))$ (where $!_{\mathsf{Y}}$ is the unique morphism $\mathsf{Y} \longrightarrow \mathsf{T}_B$) and $\mathsf{v} \longmapsto f(\mathsf{v})$. This is another property passed on to arbitrary slicing sites. Explicitly, for a slicing site $(\mathscr{A}, P)$ any morphism $f : B \longrightarrow A$ in $\mathscr{A}$ can be factored as (3), where $\mathsf{X} = Pf(\mathsf{T}_{PB})$, and f_{T} is the morphism determined by this very factorization and the equation $\varepsilon^A_{\mathsf{X}}{\cdot}Pf_{\mathsf{T}} = (Pf)_{\mathsf{T}}$. (In either case we can alternatively define f_{T} by putting $f_{\mathsf{T}} := (f{\downarrow}\mathsf{T}_B){\cdot}(\delta^{PB}_{\mathsf{T}_B})^{-1}$.) We notice that the right factor δ^A_{X} is a discrete fibration and the left factor f_{T} is repr. preserving. So every morphism in a slicing site can be factored into a repr.-preserving morphism and a discrete fibration.

Do the class of repr.-preserving morphisms and the class of discrete fibrations form a factorization system? No, not in general: an invertible functor between categories with more than one terminal object cannot map every potential repr. of the domain to every potential repr. of the range; and so $(\mathrm{F6})_{\mathrm{l}}$ fails to hold true in $\check{\mathbf{C}}\mathbf{at}$. This said, we can still consider the remaining entries in our list. Conditions (F1), (F2), (F3) and $(\mathrm{F6})_{\mathrm{r}}$ are easily verified. The observation that (F4) holds true has initiated this discussion. This leaves us with (F5). Here we base our verification on a fact that comes in handy at other points of this work as well.

1.1. Lemma. *Let $(\mathscr{A}, P)$ be a category over* $\mathbf{Cat}$. *Let f and r be two morphisms in $\mathscr{A}$ with common range, the latter a discrete fibration:*

$$\begin{array}{ccc} & & B \\ & \overset{g}{\nearrow} & \downarrow r \\ C & \xrightarrow[f]{} & A. \end{array} \tag{4}$$

Let Y be an object of PB, and let Z be a terminal object of PC satisfying $Pf(\mathsf{Z}) = Pr(\mathsf{Y})$ in PA. There is precisely one morphism g as shown, rendering the triangle commutative and satisfying $Pg(\mathsf{Z}) = \mathsf{Y}$.

We foremost have in mind the situation in which $(\mathscr{A}, P)$ is a category over $\check{\mathbf{C}}\mathbf{at}$ and $\mathsf{Z} = \mathsf{T}_{PC}$.

Proof. The statement with the premiss that $(\mathscr{A}, P)$ be a category with slicing is proved in [9]. Apply this variant to the image of (4) in $\mathbf{Cat}$ and then use the cartesianness of r. ■

The lemma has a symmetric variant, obtained by adding to the premiss that also f be a discrete fibration and also Y be terminal and to the conclusion that (the unique) g is invertible. Here is a neat little consequence.

1.2. COROLLARY. *Let $(\mathscr{A}, P)$ be a category over $\check{\mathbf{C}}\mathbf{at}$. A morphism $f : C \longrightarrow A$ in $\mathscr{A}$ is the projection associated with a slice object of objects over an object X of PA if and only if f is a discrete fibration with $Pf(\mathsf{T}_{PC}) = \mathsf{X}$. The skewness is uniquely determined.*

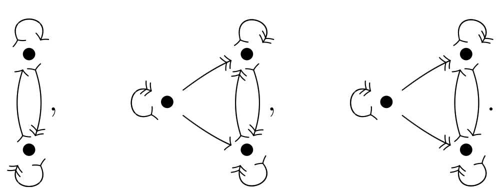
PROOF. The necessity of the condition is clear. As for sufficiency and uniqueness of the skewness, apply the symmetric variant of the lemma to Pf and δ_{X}^{PA}. ■

We are now ready to verify that a slicing site $(\mathscr{A}, P)$ satisfies (F5). Consider a commutative square (1) in $\mathscr{A}$. Put $\mathsf{Y} = Ph(\mathsf{T}_{PD})$. Since $Pf(\mathsf{T}_{PC}) = Pf{\cdot}Pl(\mathsf{T}_{PD}) = Pr{\cdot}Ph(\mathsf{T}_{PD})$, we can apply the lemma (existence) to f to obtain $g : C \longrightarrow B$ such that $r{\cdot}g = f$ and $Pg(\mathsf{T}_{PC}) = \mathsf{Y}$. Since then $r{\cdot}(g{\cdot}l) = f{\cdot}l = r{\cdot}h$ and $P(g{\cdot}l)(\mathsf{T}_{PD}) = Pg(\mathsf{T}_{PC}) = \mathsf{Y} = Ph(\mathsf{T}_{PD})$, we can apply the lemma (uniqueness) to $f{\cdot}l = r{\cdot}h$ to obtain $g{\cdot}l = h$. Now suppose that $\tilde{g}$ satisfies $\tilde{g}{\cdot}l = h$ and $r{\cdot}\tilde{g} = f$. The former equality implies $P\tilde{g}(\mathsf{T}_{PC}) = \mathsf{Y}$, which together with the latter shows that we can apply the lemma (uniqueness) to f to obtain $\tilde{g} = g$.

Semirepleteness

So the negative answer to our question can be given a positive supplement: apart from $(\mathrm{F6})_{\mathrm{l}}$ all the above conditions on a factorization system are satisfied by the repr.-preserving morphisms and the discrete fibrations in a slicing site. The search for a vocabulary to describe what we are seeing has convinced me of the need to redefine the term already in use (thus extorting an outright positive answer). *In the present work, we take 'factorization system' to be axiomatized merely by conditions* (F1–5). We drop the entire condition (F6) for self-duality. A factorization system satisfying $(\mathrm{F6})_{\mathrm{l}}$ or $(\mathrm{F6})_{\mathrm{r}}$ will be called *left semireplete* or *right semireplete*, respectively. A factorization system in the standard sense (that is, satisfying $(\mathrm{F6})_{\mathrm{l}}$ *and* $(\mathrm{F6})_{\mathrm{r}}$) will be called *replete*.

In the list of axioms just made official, no entry is entirely superfluous. In each of the six instances of a closure condition, there is a counterexample that in a certain sense is universal. The ones for $(\mathrm{F1})_{\mathrm{l}}$, $(\mathrm{F2})_{\mathrm{l}}$, $(\mathrm{F3})_{\mathrm{l}}$ are the categories with underlying graphs



We now have to distinguish between the two morphism classes $\mathscr{A}^{\times}$ and $\mathcal{L} \cap \mathcal{R}$. As the reader will have guessed, we call the elements of the latter *both-sided morphisms* and represent them graphically by the symbol '$\rightarrowtail\!\!\twoheadrightarrow$' (as has already happened). The argument showing that $\mathcal{L} \cap \mathcal{R} \subseteq \mathscr{A}^{\times}$ still goes through. Moreover, by the respective halves of (F1) and (F3) $\mathcal{L}$ and $\mathcal{R}$, and therefore $\mathcal{L}\cap\mathcal{R}$, are closed with respect to inversion. So $\mathcal{L}\cap\mathcal{R}$ is an all-object subgroupoid of $\mathscr{A}$. By (F3) a morphism g as in the conclusion of (F5)$'$ has to be both-sided; so $(\mathcal{L},\mathcal{R})$-factorizations are unique up to unique *both-sided* morphisms. This variant of (F5)$'$ on its own can in fact replace (F3) and (F5) together in our list of axioms. (In conjunction with $(\mathrm{F1})_\mathrm{r}$ and $(\mathrm{F1})_\mathrm{l}$ it implies $(\mathrm{F3})'_\mathrm{l}$ and $(\mathrm{F3})'_\mathrm{r}$, respectively.)

The inclusions ${}^{\perp}\mathcal{L}\subseteq\mathcal{R}$ and $\mathcal{R}^{\perp}\subseteq\mathcal{L}$ are now false in general. Their standard proof gets stuck after establishing ${}^{\perp}\mathcal{L}\subseteq\mathcal{R}\cdot(\mathscr{A}^{\times}\cap\mathcal{L})$ and $\mathcal{R}^{\perp}\subseteq(\mathscr{A}^{\times}\cap\mathcal{R})\cdot\mathcal{L}$. The converses of these two inclusions quite clearly hold true, so that in fact

$$ {}^{\perp}\mathcal{L}=\mathcal{R}\cdot(\mathscr{A}^{\times}\cap\mathcal{L}) \quad\text{and}\quad \mathcal{R}^{\perp}=(\mathscr{A}^{\times}\cap\mathcal{R})\cdot\mathcal{L}. \tag{5}$$

We call the elements of $\mathcal{R}^{\perp}$ *weakly left morphisms* and the elements of ${}^{\perp}\mathcal{L}$ *weakly right morphisms*. We have all but seen that also $(\mathcal{R}^{\perp},\mathcal{R})$, $(\mathcal{L},{}^{\perp}\mathcal{L})$ and $(\mathcal{R}^{\perp},{}^{\perp}\mathcal{L})$ are factorization systems; more particularly, a left-semireplete, a right-semireplete and a replete one, respectively. We may call them the *left semirepletion*, the *right semirepletion* and the *repletion* of $(\mathcal{L},\mathcal{R})$.

Let us briefly look at a special case. For two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms in $\mathscr{A}$, any two of the conditions '$\mathcal{R}=\mathscr{A}$', '$\mathcal{L}$ is an all-object subgroupoid' and '$(\mathcal{L},\mathcal{R})$ is a right-semireplete factorization system' imply the third. If they all hold true, then the (left semi-)repletion of $(\mathcal{L},\mathcal{R})$ is $(\mathscr{A}^{\times},\mathscr{A})$. (Proofs: exercise.) A noteworthy instance of such an $(\mathcal{L},\mathcal{R})$ is $(\mathscr{A}^{0},\mathscr{A})$.

We have seen that in a slicing site, the repr.-preserving morphisms and the discrete fibrations are the left morphisms and the right morphisms, respectively, for a right-semireplete factorization system. Furthermore we can easily check that the associated weakly left morphisms are the weakly repr.-preserving ones. Now, in dealing with the repletion, we can ignore repr.s. In other words, instead of $\check{\mathbf{C}}\mathbf{at}$ we may as well consider the full subcategory of **Cat** whose objects are those categories that merely possess terminal objects. Let us denote this by $\mathbf{Cat}_{(\times)}$ (following a pattern that will become clearer in section 3). We infer that $\mathbf{Cat}_{(\times)}$ carries a replete factorization system whose right morphisms are the discrete fibrations and whose left morphisms are those functors that preserve terminality. This is in fact induced by a replete factorization system on the whole of **Cat**, namely the one today known as the *comprehensive* factorization system (and which had its first public appearance, in the form of its opposite, in [13]): here the right morphisms are the discrete fibrations and the left morphisms are those functors that are final.

The present concept of a factorization system is perhaps the most reasonable common generalization of the standard one and that of a *strict* factorization system, which was introduced in [4]. Whereas the former has the single additional requirement that the all-object subgroupoid $\mathcal{L}\cap\mathcal{R}$ of $\mathscr{A}$ be the largest one, $\mathscr{A}^{\times}$, the latter has the single additional requirement that that all-object subgroupoid be the smallest one, $\mathscr{A}^{0}$. We can safely retain the latter's name, strictness now being a possible property of a factorization system.

Remark. It is interesting to see how the same generalization can be achieved with certain higher-level formulations of the notions. My main reference here is the article [11]. It shows strict factorization systems (on small categories) to amount to

- distributive laws of monads in the bicategory of (small) matrices with entries in **Set**. In its last section it goes on to describe an adaption of this concept that is accordingly related to replete factorization systems. To obtain arbitrary factorization systems we can simply drop the last of the three requirements imposed there.
- strict algebras for the strict monad $()^{\mathbf{2}}$ on the 2-category **Cat** (of which in this instance only the underlying 1-category is at issue). Shown in the earlier work [7] was that, in the same manner, replete factorization systems amount to weak algebras for the same monad. Arbitrary factorization systems in this manner amount to T-weak algebras for the strict monad $()^{\mathbf{2}}$ on the locally stratified 2-category **Str Cat**. (See the last section of the present work to make sense of this statement.)

Semistrictness

We now know that slicing-site structures induce right-semireplete factorization systems. What about the converse? That is, is every right-semireplete factorization system induced by a slicing-site structure? Again the answer starts disappointingly: no, it is not. To see why, consider the following condition.

(F7)$_{\mathrm{l}}$ In a situation $C \overset{l_0}{\underset{l_1}{\rightrightarrows}} B \overset{r}{\rightarrowtail} A$, if $r{\cdot}l_0 = r{\cdot}l_1$, then $l_0 = l_1$.

Suppose we have an arbitrary factorization system $(\mathcal{L}, \mathcal{R})$. The premiss of (F7)$_{\mathrm{l}}$ amounts to a commutative square

$$\begin{array}{ccc} C & \overset{l_0}{\twoheadrightarrow} & B \\ {\scriptstyle l_1}\downarrow & \overset{\beta}{\nearrow} & \downarrow{\scriptstyle r} \\ B & \underset{r}{\rightarrowtail} & A, \end{array}$$

to which (F5)$'$ and (F3) apply to yield β as shown, rendering the two triangles commutative. To obtain the conclusion of (F7)$_{\mathrm{l}}$ from here, all we need is for β to be the identity. Thus condition (F7)$_{\mathrm{l}}$ is equivalent to a weaker variant of itself:

(F7)$'_{\mathrm{l}}$ For a right morphism $r : B \rightarrowtail A$ and a both-sided automorphism β of B, if $r{\cdot}\beta = r$, then $\beta = 1_B$.

Applying the uniqueness part of the lemma with $\mathsf{Y} = \mathsf{T}_{PB}$, we see that (F7)$_{\mathrm{l}}$ holds true for slicing sites (in fact, for arbitrary categories over **Čat**). But (F7)$_{\mathrm{l}}$ does not hold true for arbitrary categories with right-semireplete, or even replete, factorization systems. In fact, under just (F1–5) there is a universal counterexample which also satisfies (F6): objects are A, B, C; morphisms are $1_A : A \rightarrowtail\!\!\!\!\!\twoheadrightarrow A$, $r : B \rightarrowtail A$, $g : C \longrightarrow A$, $\beta^s : B \rightarrowtail\!\!\!\!\!\twoheadrightarrow B$ ($s \in \mathbf{Z}$), $l_s : C \twoheadrightarrow B$ ($s \in \mathbf{Z}$), $1_C : C \rightarrowtail\!\!\!\!\!\twoheadrightarrow C$; composition is such that 1_A, 1_C are neutral and that $r{\cdot}\beta^s = r$, $r{\cdot}l_s = g$, $\beta^t{\cdot}\beta^s = \beta^{s+t}$, $\beta^t{\cdot}l_s = l_{s-t}$.

A counterexample "occurring in nature" is $\mathbf{Cat}_{(\times)}$ with the factorization system of terminality-preserving functors and discrete fibrations. (See above.) Take A to be the category presented by

$$\mathsf{i} \circlearrowright \mathsf{X} \xrightarrow{\mathsf{x}} \mathsf{T}, \quad \mathsf{i} \circ \mathsf{i} = 1_{\mathsf{X}}, \quad \mathsf{x} \circ \mathsf{i} = \mathsf{x}$$

(the cyclic group of order 2, suspended, and a terminal object added). Take B to be the slice category of objects over X, which is presented by

$$\begin{array}{c} (\mathsf{X}, 1_{\mathsf{X}}) \\ {\scriptstyle \mathsf{i}}\uparrow\ \ \downarrow{\scriptstyle \mathsf{i}}, \\ (\mathsf{X}, \mathsf{i}) \end{array} \qquad \mathsf{i} \circ \mathsf{i} = 1_{(\mathsf{X},*)}$$

(the 2-object homogeneous category) and r to be the associated projection. Finally, take β to be the autofunctor induced by i, which interchanges the two objects of B. Note that we have selected the objects for minimality in size; all we actually needed was for some automorphism group in A to be non-trivial.

We may render condition $(\mathrm{F7})_{\mathrm{l}}$ by saying that an $(\mathcal{L}, \mathcal{R})$-factorization is (strictly) determined by its right half. This suggests that we call such a factorization system *left semistrict*. Then the dual concept obviously receives the name '*right semistrict*'. All this fits quite well with the term 'strict' for what is the case that an individual factorization is (strictly) determined entirely. Strict factorization systems are left and right semistrict, but the converse of this statement is false.

Another way of rendering condition $(\mathrm{F7})_{\mathrm{l}}$ is to say that the right morphisms are monic relative to the left ones. Thus we instantly see that among the left-semistrict factorization systems are all those standard (replete) ones where the right morphisms are (absolutely) monic. It is not true that left semistrictness of a replete factorization system implies for the right morphisms to be monic. A counterexample dear to me is provided by the slicing site of polytopes and polytopic maps: polytopes have unique terminal objects, whence the repleteness; but, for instance, the two polytopic maps to the monogon from the line segment are discrete fibrations that are not monic, as witnessed by the two polytopic maps to the line segment from the point. (See [9], section 4.)

From the left half of (5) one easily infers that if all right morphisms are monic relative to all left ones, then so are all weakly right morphisms. Thus left semistrictness is indeed a left condition, namely in the technical sense that its satisfaction by a factorization system depends on the left-morphism class only.

For later reference we note that on any category $\mathscr{A}$ there are two extreme instances of a factorization system that is both right semireplete and left semistrict, namely $(\mathscr{A}^0, \mathscr{A})$ and $(\mathscr{A}, \mathscr{A}^\times)$.

Remark. An anonymous reviewer has pointed out to me the recent paper [2], in which the repleteness versions of $(\mathrm{F7})'_{\mathrm{l}}$ and $(\mathrm{F7})'_{\mathrm{r}}$ make explicit appearances. Its subject belongs to the realm of abstract homotopy theory. The reader will recall that a *Reedy structure* on a small category $\mathscr{A}$ allows a certain way of promoting a model-category structure from a given complete and cocomplete category $\mathscr{K}$ to the functor category $\mathscr{K}^{\mathscr{A}}$. Put in present terms, a Reedy structure is a factorization system $(\mathcal{L}, \mathcal{R})$ that is strict *and* replete (so that the only morphisms invertible in $\mathscr{A}$ are the identities), and for which there are conservative functors $\mathcal{L}^{\mathrm{op}} \longrightarrow \mathbf{\Omega}$ and $\mathcal{R} \longrightarrow \mathbf{\Omega}$ agreeing on objects ($\mathbf{\Omega}$ being the well-ordered class of all ordinals). The paper shows how the purpose stated, at least as far as cofibrantly generated model categories are concerned, is met under more general conditions, namely with strictness replaced by right semistrictness.

For the benefit of the reader I wrap up the main results obtained thus far.

1.3. THEOREM. *For a slicing site, the class of repr.-preserving morphisms and the class of discrete fibrations form a left-semistrict right-semireplete factorization system.* ■

The converse

If we adjust our question above to take left semistrictness into account, the answer becomes positive. Explicitly: any left-semistrict right-semireplete factorization system is induced by a slicing-site structure. We now go about proving this fact.

Let $\mathscr{A}$ be a category equipped with a left-semistrict right-semireplete factorization system $(\mathcal{L}, \mathcal{R})$. We are going to construct a functor $P : \mathscr{A} \longrightarrow \check{\mathbf{C}}\mathbf{at}$ that will turn out to possess all the desired properties.

Let A be an object of $\mathscr{A}$. We consider the category $\mathcal{R}{\downarrow}A$, the slice category over the right-morphism category $\mathcal{R}$; by $(\mathrm{F3})_{\mathrm{r}}$ it is full as a subcategory of $\mathscr{A}{\downarrow}A$, the slice category over $\mathscr{A}$ itself. We further consider both-sided isomorphism as an equivalence relation among the objects of $\mathcal{R}{\downarrow}A$. (Two objects (X_0, x_0) and (X_1, x_1) are both-sidedly isomorphic if there is a both-sided isomorphism ξ in $\mathscr{A}$ rendering the triangle

$$\begin{array}{ccc} X_0 & \xrightarrow{x_0} & \\ {\scriptstyle \xi}\downarrow & & A \\ X_1 & \xrightarrow{x_1} & \end{array} \tag{6}$$

commutative.) We choose a representative system and form the associated full subcategory. (In other words, we choose a skeleton with respect to both-sided isomorphism.) We define PA to be this category, represented by the only both-sided morphism among its objects. We use brackets to indicate that an object or morphism of $\mathcal{R}{\downarrow}A$ is considered as an object or morphism of PA. So a typical object of PA appears as $[X, x]$ (where $X \in \mathscr{A}$ and $x : X \rightarrowtail A$ chosen), and a typical morphism $[X', x'] \longrightarrow [X, x]$ in PA appears as $[u]$ (where $u : X' \rightarrowtail X$ with $x{\cdot}u = x'$).

(The foundationally minded reader will have noticed that this "construction" makes blatant use of the axiom of choice. We could have avoided this in a fairly obvious way by expressing PA as a quotient rather than a part of $\mathcal{R}{\downarrow}A$: by left semistrictness, the ξ as of (6) are uniquely determined; that is, the both-sided isomorphisms form an essentially discrete all-object subgroupoid and can therefore be "divided out" by. The disadvantages of this more cautious approach are the necessity to deal head-on with those issues that here fall under the caption 'well-definedness' and, clinching the matter, the lack of a generalization as described in section 3.)

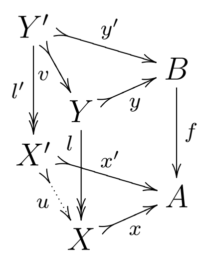
The totality of choices made here will become a global choice of slice objects. In the present context we refer to it as a *global choice of right factors*.

Among all factorizations $r{\cdot}l$ of a given morphism to A that exist by (F4), there is precisely one for which r appears as an object of PA. Let us call it the *preferred* factorization. (We have uniqueness of r by construction and, then, uniqueness of l by left semistrictness.)

Now let $f : B \longrightarrow A$ be a morphism in $\mathscr{A}$; we wish to exhibit a functor $Pf : PB \longrightarrow PA$. For an object $[Y, y]$ of PB, take the preferred factorization $x{\cdot}l$ of $f{\cdot}y$, visualized by the commutative square

$$\begin{array}{ccc} Y & \overset{y}{\rightarrowtail} & B \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle f} \\ X & \underset{x}{\rightarrowtail} & A, \end{array} \tag{7}$$

and put $Pf([Y, y]) := [X, x]$. For a morphism $[v] : [Y', y'] \longrightarrow [Y, y]$, take the unique morphism u rendering the wedge



(8)

commutative, and put $Pf([v]) := [u]$. Clearly these specifications make Pf a functor.

We now have to establish that P itself is a functor $\mathscr{A} \longrightarrow \check{\mathbf{C}}\mathbf{at}$. We start by examining the action of the images on objects. For the nullary instance, let $A \in \mathscr{A}$ and $[X, x] \in PA$. The preferred factorization of $1_A{\cdot}x$ has to be $x{\cdot}1_X$; thus $P1_A([X, x]) = [X, x]$. For the binary instance, let $C \xrightarrow{h} B \xrightarrow{f} A \in \mathscr{A}$ and $[Z, z] \in PC$. Consider the preferred factorizations $y{\cdot}n$ of $h{\cdot}z$ and $x{\cdot}l$ of $f{\cdot}y$, visualized by the stack of two commutative squares

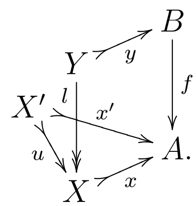
$$\begin{array}{ccc} Z & \overset{z}{\rightarrowtail} & C \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \overset{y}{\rightarrowtail} & B \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle f} \\ X & \underset{x}{\rightarrowtail} & A. \end{array} \tag{9}$$

The preferred factorization of $(f{\cdot}h){\cdot}z$ has to be $x{\cdot}(l{\cdot}n)$; thus $P(f{\cdot}h)([Z, z]) = [X, x] = Pf([Y, y]) = Pf{\cdot}Ph([Z, z])$. The excess information, concerning the left halves of the preferred factorizations in question, has its use in the examination of the action on morphisms, which, this said, can be considered done.

Before we investigate whether the functor P constitutes a slicing-site structure, we show that it induces the correct morphism classes. We can readily see that the left morphisms are precisely the repr.-preserving ones: if in (7) y is left, then so is x precisely if so is f. More work is involved in showing that the right morphisms are precisely the discrete fibrations.

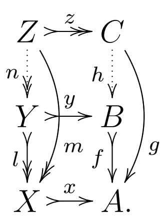
As for the forward implication, suppose that $f : B \longrightarrow A$ is a right morphism.

We first show that Pf is a discrete fibration. So let $[Y, y] \in PB$, and let $[u] : [X', x'] \longrightarrow [X, x] = Pf([Y, y]) \in PA$:



We have to show that there is a unique way of completing this figure to become (8). Suppose temporarily the completion carried out. Since f is right, l and l' are both-sided. So $v = l^{-1}{\cdot}u{\cdot}l'$, and $y'{\cdot}l'^{-1}$ is the preferred factorization of $y{\cdot}l^{-1}{\cdot}u$. This verifies uniqueness, and it yields the construction that all but immediately verifies existence.

Now we show that f is cartesian with respect to P. Let $g : C \longrightarrow A \in \mathscr{A}$, and let $h' : PC \longrightarrow PB \in \check{\mathbf{C}}\mathbf{at}$ with $Pf{\cdot}h' = Pg$. Consider the repr. $\mathsf{T}_{PC} =: [Z, z]$ of PC. Put $h'(\mathsf{T}_{PC}) =: [Y, y]$; then $Pf([Y, y]) = Pg(\mathsf{T}_{PC}) =: [X, x]$:



Let $h : C \longrightarrow B$ with $Ph = h'$ and $f{\cdot}h = g$. The former equation, applied to T_{PC}, yields $n \in \mathcal{L}$ with $y{\cdot}n = h{\cdot}z$, and from the latter equation we further infer using left semistrictness that $l{\cdot}n = m$. Thus $n = l^{-1}{\cdot}m$ and $h = y{\cdot}n{\cdot}z^{-1}$. This verifies uniqueness. Conversely, suppose n and h given by these equations. Then firstly $f{\cdot}h = x{\cdot}m{\cdot}z^{-1} = g$. Secondly $Ph(\mathsf{T}_{PC}) = [Y, y]$, and applying the lemma (uniqueness) to $\check{\mathbf{C}}\mathbf{at}$ we obtain $Ph = h'$. This verifies existence.

As for the backward implication, suppose that $g : C \longrightarrow A$ is a discrete fibration. Consider its preferred factorization $C \xrightarrow{h} B \xrightarrow{f} A$. By what we have already shown, f is a discrete fibration as well, and h is repr. preserving, whence $Pg(\mathsf{T}_{PC}) = Pf(\mathsf{T}_{PB})$. We now apply the symmetrization of the lemma to find that h is invertible. Thus h is a right morphism, and hence so is $f{\cdot}h = g$.

We can now quickly show that $(\mathscr{A}, P)$ is indeed a slicing site. Let A be an object of $\mathscr{A}$, and let $[X, x]$ be an object of PA. By the corollary to the lemma all we have to do is to exhibit a discrete fibration $f : B \rightarrowtail A$ such that $Pf(\mathsf{T}_{PB}) = [X, x]$. Here it is: take $B = X$ and $f = x$.

Well-poweringness

The reader will object that we have neglected the question of size: the represented categories PA have to be small in order to belong to $\check{\mathbf{C}}\mathbf{at}$. In my response I am going to go slightly afield. We are working on the basis of some form of set theory, 'small' signifying membership in a universe $\mathfrak{U}$ singled out at the beginning. Now, practically this entire paper can be reinterpreted by detaching the term 'small' from any condition (so that it becomes redundant). Then a category of all small things of a given kind, such as **Cat** or **Čat**, will not exist barring the use of proper classes; but neither does it have to, since what we wish to distinguish among are functors to it (from necessarily small categories), and these can be appropriately defined and then handled just as they are otherwise. Thus we obtain a purely structural concept of a slicing site. All size-related snags can simply be ignored; the disputed argument will go through as stated. If I have eschewed this interpretation, this is mostly because I wish to retain the correspondence with categories with slicing, which (except for the trivial one) cannot be small either. Then of course the reader's objection is valid; it will be addressed shortly.

In the meantime we note a genuine point of the foregoing considerations. We may look at both concepts of a slicing site in parallel; let us use the terms 'slicing site relative to $\mathfrak{U}$' and 'absolute slicing site' for distinction. Since $\mathfrak{U}$ is closed with respect to the formation of slice categories, a slicing site relative to $\mathfrak{U}$ is precisely an absolute slicing site whose structural functor maps into $\mathfrak{U}$. This will be useful on certain occasions, when a requisite smallness can be established via a partial 'everything small' reinterpration.

The promised treatment of the size question consists in a definition. If indeed all the PA are small we call $(\mathcal{L}, \mathcal{R})$ *well-powering*. Several remarks have to be annexed. To account for the option of relabelling elements we take 'small' in this context to mean: being isomorphic to (rather than itself being) a member of $\mathfrak{U}$. Then it is clear that we have well-definedness; the condition is equivalent to the $\mathcal{R}{\downarrow}A$ having small-many (right) morphisms between each two objects and small-many both-sided-isomorphism classes. All the statements actually involved in this definition apply to arbitrary (not just left-semistrict right-semireplete) factorization systems. A final comment concerns the name. This is based on the one in use for the paradigmatic case of a replete factorization system $(\mathcal{L}, \mathcal{R})$, namely with $\mathcal{R}$ the class of monomorphisms: here if the condition is satisfied the carrying category $\mathscr{A}$ is called well-power*ed*.

For any category $\mathscr{A}$ whatsoever the factorization system $(\mathscr{A}, \mathscr{A}^{\times})$ (for which $PA \simeq \mathbf{1}$) is well-powering. In order that all factorization systems on a given category $\mathscr{A}$ be well-powering it is sufficient, and because of the instance $(\mathscr{A}^0, \mathscr{A})$ (for which $PA \simeq \mathscr{A}{\downarrow}A$) also necessary, that $\mathscr{A}$ be *initially small*, meaning that all slice categories over $\mathscr{A}$ are small.

For $\mathscr{A}$ to be initially small it suffices that the object sets of the $\mathscr{A}{\downarrow}A$ be small, that is, that there be only small-many morphisms to each object. More generally, *for a* left-semistrict *factorization system* $(\mathcal{L}, \mathcal{R})$ *on* $\mathscr{A}$ *to be well-powering it suffices that that the object sets of the associated* PA *be small*, that is, that there be only small-many both-sided-isomorphism classes of right morphisms to each object. Let us prove this from first principles. Assuming the object set of every PA small, we have to show that every hom-set of every PA is small as well. In fact, given $A \in \mathscr{A}$ and $[X, x], [X', x'] \in PA$ we show that the evident map from the set of morphisms $[X', x'] \longrightarrow [X, x]$ to the set of objects of PX is injective: any two elements $[u_0]$ and $[u_1]$ of the former agree if qua $(X', u_0), (X', u_1) \in \mathcal{R}{\downarrow}X$

they are represented by the same element of the latter. Indeed, there then is a both-sided automorphism ξ' of X' such that $u_1{\cdot}\xi' = u_0$; hence $x'{\cdot}\xi' = x{\cdot}u_1{\cdot}\xi' = x{\cdot}u_0 = x'$; hence by left semistrictness $\xi' = 1_{X'}$; hence $u_1 = u_0$.

This subsection closes with some more assorted facts. Well-poweringness is another left condition: by the left half of (5) any object of ${}^{\perp}\mathcal{L}{\downarrow}A$ has a left isomorphism to some object of $\mathcal{R}{\downarrow}A$, so that in fact PA, up to repr.-preserving isomorphism, does not depend on the $\mathcal{R}$ accompanying $\mathcal{L}$. The factorization system associated with a slicing site (theorem 1.3) is well-powering: this will follow from the 'everything small' interpretation of the proof of theorem 1.4 (uniqueness). If a factorization system $(\mathcal{L}, \mathcal{R})$ is well-powering, and the connected components of its both-sided-isomorphism groupoid $\mathcal{L}\cap\mathcal{R}$ are small, then, and only then, its right-morphism category $\mathcal{R}$ is initially small.

Here is the whole of what we are intending to show.

1.4. Theorem. *Any well-powering left-semistrict right-semireplete factorization system is induced in the manner of theorem 1.3 by a slicing-site structure, unique up to a unique repr.-preserving (invertible) natural transformation.*

The existence part has been dealt with. Uniqueness will be established in the course of a more general examination; a strategy that will become subject to iteration.

Note that the uniqueness statement is the best one can hope for, considering the rather evident fact that if P is a slicing-site structure inducing $(\mathcal{L}, \mathcal{R})$, then so is every functor repr.-preservingly isomorphic to P.

Higher-cell lemmata

1.5. Lemma. *Let $(\mathscr{A}, P)$ be a slicing site, and let Q be an arbitrary functor $\mathscr{A} \longrightarrow$ **Čat**. There is a laxly natural transformation $P \longrightarrow Q$ that is weakly repr. preserving and weakly natural on discrete fibrations. It is unique up to a unique (invertible) modification.*

The reader will have noticed that this is the first major reference to the 2$^{\text{nd}}$ dimension of **Čat** (or **Cat** for that matter), which is made up of all pertinent natural transformations. The meaning of 'weakly repr. preserving' here is governed by the stipulation that an adjective describing a transformation and not otherwise defined is to be understood with the addition 'on objects' and thus refers to the 1-dimensional components. (So for $\chi : P \longrightarrow Q$ to be strictly or weakly repr. preserving is for all $\chi_A : PA \longrightarrow QA$ to be strictly or weakly repr. preserving, respectively.)

Proof. Having made a global choice of slice objects, we construct a transformation χ as announced.

First, let A be an object of $\mathscr{A}$; we have to exhibit a weakly repr.-preserving functor $\chi_A : PA \longrightarrow QA$. For $\mathsf{X} \in PA$, we put $\chi_A(\mathsf{X}) := Q\delta^A_{\mathsf{X}}(\mathsf{T}_{Q(A\downarrow\mathsf{X})}) \in QA$. For $\mathsf{u} : \mathsf{X}' \longrightarrow \mathsf{X} \in PA$, we put $\chi_A(\mathsf{u}) := Q\delta^A_{\mathsf{X}}(!_{Q(A\downarrow\mathsf{u})}) : Q\delta^A_{\mathsf{X}'}(\mathsf{T}_{Q(A\downarrow\mathsf{X}')}) \longrightarrow Q\delta^A_{\mathsf{X}}(\mathsf{T}_{Q(A\downarrow\mathsf{X})}) \in QA$. The functoriality of χ_A is evident. If $\mathsf{X} = \mathsf{T}_{PA}$, then δ^A_{X} is invertible, and hence $\chi_A(\mathsf{X})$ is terminal; thus, χ_A is weakly repr. preserving.

Second, let $f : B \longrightarrow A$ be a morphism in $\mathscr{A}$; we have to exhibit a natural transformation $\chi f : Qf{\cdot}\chi_B \longrightarrow \chi_A{\cdot}Pf$:

$$\begin{array}{ccc} PB & \xrightarrow{\chi_B} & QB \\ {\scriptstyle Pf}\downarrow & \overset{\chi f}{\swarrow} & \downarrow{\scriptstyle Qf} \\ PA & \xrightarrow{\chi_A} & QA. \end{array}$$

For $\mathsf{Y} \in PB$, say with Pf-image $\mathsf{X} \in PA$, we wish to define a morphism $\chi f(\mathsf{Y})$ between the two objects

$$Qf{\cdot}\chi_B(\mathsf{Y}) = Qf{\cdot}Q\delta^B_{\mathsf{Y}}(\mathsf{T}_{Q(B\downarrow\mathsf{Y})}) = Q\delta^A_{\mathsf{X}}{\cdot}Q(f\downarrow\mathsf{Y})(\mathsf{T}_{Q(B\downarrow\mathsf{Y})})$$

and

$$\chi_A{\cdot}Pf(\mathsf{Y}) = \chi_A(\mathsf{X}) = Q\delta^A_{\mathsf{X}}(\mathsf{T}_{Q(A\downarrow\mathsf{X})});$$

we do so by putting $\chi f(\mathsf{Y}) := Q\delta^A_{\mathsf{X}}(!_{Q(f\downarrow\mathsf{Y})})$. For $\mathsf{v} : \mathsf{Y}' \longrightarrow \mathsf{Y} \in PB$, say with Pf-image $\mathsf{u} : \mathsf{X}' \longrightarrow \mathsf{X} \in PA$, we have

$$\begin{aligned} \chi_A{\cdot}Pf(\mathsf{v}) \circ \chi f(\mathsf{Y}') &= \chi_A(\mathsf{u}) \circ Q\delta^A_{\mathsf{X}'}(!_{Q(f\downarrow\mathsf{Y}')}) \\ &= Q\delta^A_{\mathsf{X}}(!_{Q(A\downarrow\mathsf{u})}) \circ Q\delta^A_{\mathsf{X}}{\cdot}Q(A\downarrow\mathsf{u})(!_{Q(f\downarrow\mathsf{Y}')}) \\ &= Q\delta^A_{\mathsf{X}}(!_{Q(A\downarrow\mathsf{u}){\cdot}Q(f\downarrow\mathsf{Y}')}) \\ &= Q\delta^A_{\mathsf{X}}(!_{Q(f\downarrow\mathsf{Y}){\cdot}Q(B\downarrow\mathsf{v})}) \\ &= Q\delta^A_{\mathsf{X}}(!_{Q(f\downarrow\mathsf{Y})}) \circ Q\delta^A_{\mathsf{X}}{\cdot}Q(f\downarrow\mathsf{Y})(!_{Q(B\downarrow\mathsf{v})}) \\ &= Q\delta^A_{\mathsf{X}}(!_{Q(f\downarrow\mathsf{Y})}) \circ Qf{\cdot}Q\delta^B_{\mathsf{Y}}(!_{Q(B\downarrow\mathsf{v})}) \\ &= \chi f(\mathsf{Y}) \circ Qf{\cdot}\chi_B(\mathsf{v}); \end{aligned}$$

this shows naturality. If f is a discrete fibration, then $f\downarrow\mathsf{Y}$ is invertible, hence so is $!_{Q(f\downarrow\mathsf{Y})}$, hence so is $\chi f(\mathsf{Y})$.

Third and last, we have to verify compatibility with composition preservation. As for the nullary instance, let A be an object of $\mathscr{A}$. For $\mathsf{X} \in PA$ we find

$$1_{\chi_A}(\mathsf{X}) = 1_{\chi_A(\mathsf{X})} = 1_{Q\delta^A_{\mathsf{X}}(\mathsf{T}_{Q(A\downarrow\mathsf{X})})} = Q\delta^A_{\mathsf{X}}(!_{Q1_{A\downarrow\mathsf{X}}}) = Q\delta^A_{\mathsf{X}}(!_{Q(1_A\downarrow\mathsf{X})}) = \chi 1_A(\mathsf{X}).$$

As for the binary instance, let $C \xrightarrow{h} B \xrightarrow{f} A$ be a composable pair of morphisms in $\mathscr{A}$. We have to show that

$$\begin{array}{ccc} PC & \xrightarrow{\chi_C} & QC \\ {\scriptstyle Ph}\downarrow & \overset{\chi h}{\swarrow} & \downarrow{\scriptstyle Qh} \\ PB & \xrightarrow{\chi_B} & QB \\ {\scriptstyle Pf}\downarrow & \overset{\chi f}{\swarrow} & \downarrow{\scriptstyle Qf} \\ PA & \xrightarrow{\chi_A} & QA \end{array} \quad = \quad \begin{array}{ccc} PC & \xrightarrow{\chi_C} & QC \\ {\scriptstyle P(f\cdot h)}\Bigg\downarrow & \overset{\chi(f\cdot h)}{\swarrow} & \Bigg\downarrow{\scriptstyle Q(f\cdot h)} \\ PA & \xrightarrow{\chi_A} & QA. \end{array}$$

For $\mathsf{Z} \in PC$ we set $Ph(\mathsf{Z}) =: \mathsf{Y}$ and $Pf(\mathsf{Y}) =: \mathsf{X}$ and find

$$\begin{aligned}(\chi f{\cdot}Ph \circ Qf{\cdot}\chi h)(\mathsf{Z}) &= \chi f{\cdot}Ph(\mathsf{Z}) \circ Qf{\cdot}\chi h(\mathsf{Z}) \\ &= \chi f(\mathsf{Y}) \circ Qf{\cdot}Q\delta^B_{\mathsf{Y}}(!_{Q(h\downarrow\mathsf{Z})}) \\ &= Q\delta^A_{\mathsf{X}}(!_{Q(f\downarrow\mathsf{Y})}) \circ Q\delta^A_{\mathsf{X}}{\cdot}Q(f\downarrow\mathsf{Y})(!_{Q(h\downarrow\mathsf{Z})}) \\ &= Q\delta^A_{\mathsf{X}}(!_{Q(f\downarrow\mathsf{Y})\cdot Q(h\downarrow\mathsf{Z})}) \\ &= Q\delta^A_{\mathsf{X}}(!_{Q(f\cdot h\downarrow\mathsf{Z})}) \\ &= \chi(f{\cdot}h)(\mathsf{Z}).\end{aligned}$$

We now encounter the "iteration" mentioned above.

1.6. Lemma. *Let $(\mathscr{A}, P)$ be a slicing site; let Q be a functor $\mathscr{A} \longrightarrow \check{\mathbf{C}}\mathbf{at}$; let ϕ and ψ be two laxly natural transformations $P \longrightarrow Q$, the former weakly natural on discrete fibrations, the latter weakly repr. preserving. There is a unique modification $\phi \longrightarrow \psi$.*

The symmetric variant of this lemma (with respect to ϕ and ψ) is the uniqueness part of the one we are proving. ∎

Proof of lemma 1.6. We first show uniqueness; hence we obtain the construction to show existence.

Let $\mathsf{m} : \phi \longrightarrow \psi$ be a modification; we wish to exhibit a formula expressing its components in terms not involving itself. Let A be an object of $\mathscr{A}$. Since by assumption ψ_A is weakly repr. preserving, $\psi_A(\mathsf{T}_{PA})$ is terminal in QA, and so there is precisely one morphism $\phi_A(\mathsf{T}_{PA}) \longrightarrow \psi_A(\mathsf{T}_{PA})$; thus $\mathsf{m}_A(\mathsf{T}_{PA})$ is determined. To have some handy notation, let us, for morphisms f_0 and f_1 in $\check{\mathbf{C}}\mathbf{at}$ with the same range and with the latter weakly repr. preserving, put $(!_{f_1})^{-1} =: !^{f_1}$ and $!^{f_1} \circ !_{f_0} =: !^{f_1}_{f_0}$; we just saw that $\mathsf{m}_A(\mathsf{T}_{PA}) = !^{\psi_A}_{\phi_A}$. An arbitrary object $\mathsf{X} \in PA$ comes about as $P\delta^A_{\mathsf{X}}(\mathsf{T}_{P(A\downarrow\mathsf{X})})$. The compatibility-with-naturality condition on m, applied to $\delta^A_{\mathsf{X}} : A\downarrow\mathsf{X} \longrightarrow A$, reads

$$\mathsf{m}_A{\cdot}P\delta^A_{\mathsf{X}} \circ \phi\delta^A_{\mathsf{X}} = \psi\delta^A_{\mathsf{X}} \circ Q\delta^A_{\mathsf{X}}{\cdot}\mathsf{m}_{A\downarrow\mathsf{X}}.$$

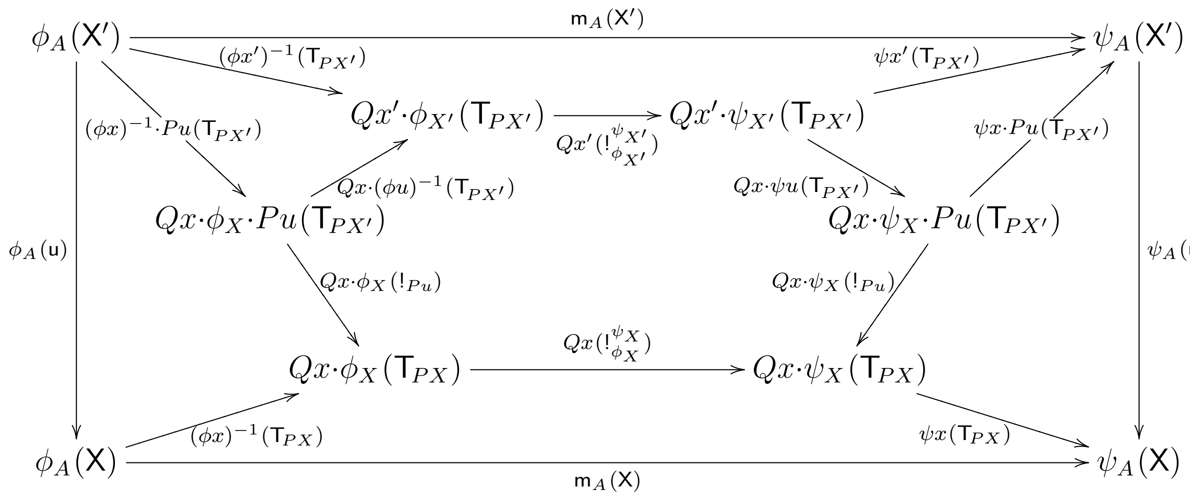
We precompose with the inverse of $\phi\delta^A_{\mathsf{X}}$, which exists by assumption, and evaluate at $\mathsf{T}_{P(A\downarrow\mathsf{X})}$, to obtain

$$\mathsf{m}_A(\mathsf{X}) = \psi\delta^A_{\mathsf{X}}(\mathsf{T}_{P(A\downarrow\mathsf{X})}) \circ Q\delta^A_{\mathsf{X}}(!^{\psi_{A\downarrow\mathsf{X}}}_{\phi_{A\downarrow\mathsf{X}}}) \circ (\phi\delta^A_{\mathsf{X}})^{-1}(\mathsf{T}_{P(A\downarrow\mathsf{X})}). \tag{10}$$

Thus we have indeed "solved for m".

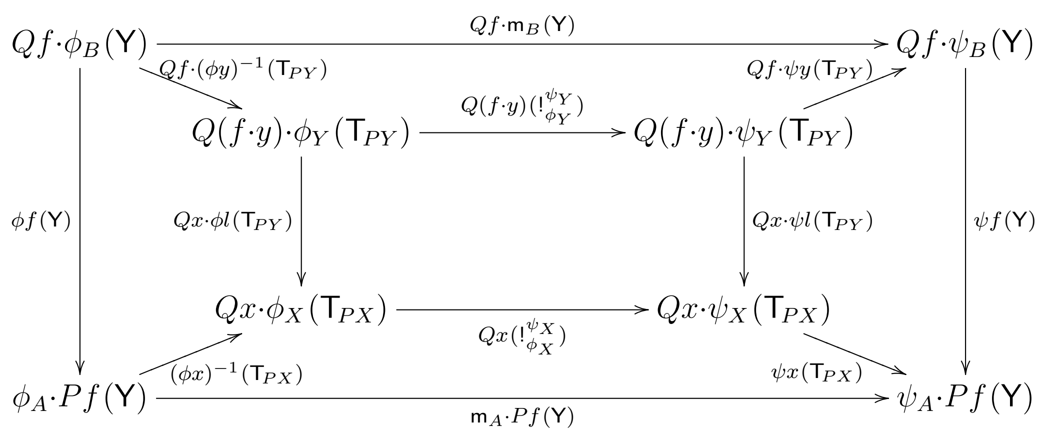
Conversely, let $\mathsf{m}_A(\mathsf{X}) : \phi_A(\mathsf{X}) \longrightarrow \psi_A(\mathsf{X})$ be given by (10).

We first have to show that each m_A is a natural transformation $\phi_A \longrightarrow \psi_A$. An arbitrary morphism $\mathsf{u} : \mathsf{X}' \longrightarrow \mathsf{X} \in PA$ comes about as $P\delta^A_{\mathsf{X}}(!_{P(A\downarrow\mathsf{u})})$. We have to verify commutativity of the outer rectangle of the diagram



in QA, where we use the abbreviations $(A{\downarrow}\mathsf{X}, \delta^A_{\mathsf{X}}) =: (X, x)$, $(A{\downarrow}\mathsf{X}', \delta^A_{\mathsf{X}'}) =: (X', x')$ and $A{\downarrow}\mathsf{u} =: u$. The top and bottom quadrangles commute by definition of m_A; the left and right quadrangles commute by naturality of, respectively, ϕx and ψx; the top left and right triangles commute by compatibility of, respectively, ϕ and ψ with composition preservation; and the inner hexagon commutes since both paths yield $Qx(!^{\psi_X}_{\phi_X \cdot Pu})$.

Now we have to show that m as a whole is compatible with the (lax) naturality of ϕ and ψ. So let $f : B \longrightarrow A$ be a morphism in $\mathscr{A}$, and let Y be an object of PB. We have to verify commutativity of the outer rectangle of the diagram



in QA, where we use the abbreviations $(B{\downarrow}\mathsf{Y}, \delta^B_{\mathsf{Y}}) =: (Y, y)$, $(A{\downarrow}Pf(\mathsf{Y}), \delta^A_{Pf(\mathsf{Y})}) =: (X, x)$ and $f{\downarrow}\mathsf{Y} =: l$. The top and bottom quadrangles commute by definition of m; the left and right quadrangles commute by compatibility of, respectively, ϕ and ψ with composition preservation; and the inner rectangle commutes since both paths yield $Qx(!^{\psi_X}_{Ql \cdot \phi_Y})$. ∎

Addenda to lemma 1.5

According to the uniqueness part of lemma 1.5 the transformation χ constructed in the proof of the existence part is "weakly independent" of the global choice of slice objects made there. The reader will have noticed that we could have reduced the remaining leeway by insisting that this choice be normal: then χ would have come out *strictly* repr. preserving. However, while this refinement would indeed strengthen the result, it would also destroy the "symmetry" between the two properties of χ. The point will be made clearer by the following addendum.

1.7. Proposition. *For the construction in the proof of lemma 1.5, the following six conditions are equivalent.*

(i) *Q respects repr.-preserving isomorphisms.*
(ii) *χ is (strictly) independent of the global choice of slice objects.*
(iii) *χ is (strictly) repr. preserving for all global choices of slice objects.*
(iv) *χ is (strictly) natural on discrete fibrations for all global choices of slice objects.*
(v) *χ is repr. preserving and natural on discrete fibrations for some global choice of slice objects.*
(vi) *There is some laxly natural transformation $P \longrightarrow Q$ that is repr. preserving and natural on discrete fibrations.*

Moreover, the transformation of (vi) *is unique (and therefore equal to χ).*

Proof. The equivalence segment follows the scheme

$$\begin{array}{ccccc} \text{(iii)} & & \text{(iv)} & & \\ & \nwarrow\!\searrow & & \searrow & \\ & & \text{(i)} \quad \nearrow & & \text{(v)} \\ & \nearrow\!\swarrow & & \nwarrow \quad \swarrow & \\ \text{(ii)} & & \text{(vi)} & & \end{array} \tag{11}$$

clockwise, starting and finishing inside the bottom notch. Note that since every functor respects isomorphisms, in order for (i) to hold true it suffices that the image of a repr.-preserving isomorphism under Q is itself repr. preserving.

(i) $\Rightarrow$ (ii). As we are going to see, the assumption yields (iii) and (iv), and these two properties determine χ.

(ii) $\Rightarrow$ (i). Let $x : X \rightarrowtail\!\!\!\twoheadrightarrow A$ be a repr.-preserving isomorphism. Make two global choices of slice objects, one with $(A{\downarrow}\mathsf{T}_{PA}, \delta^A_{\mathsf{T}_{PA}}) = (X, x)$, one with $(A{\downarrow}\mathsf{T}_{PA}, \delta^A_{\mathsf{T}_{PA}}) = (A, 1_A)$. By assumption they lead to the same laxly natural transformation χ, and so $Qx(\mathsf{T}_{QX}) = \chi_A(\mathsf{T}_{PA}) = \mathsf{T}_{QA}$.

(i) $\Rightarrow$ (iii). $\delta^A_{\mathsf{T}_{PA}}$ is a repr.-preserving isomorphism, hence by assumption $Q\delta^A_{\mathsf{T}_{PA}}$ is repr. preserving, hence $\chi_A(\mathsf{T}_{PA}) = Q\delta^A_{\mathsf{T}_{PA}}(\mathsf{T}_{Q(A{\downarrow}\mathsf{T}_{PA})}) = \mathsf{T}_{QA}$.

(iii) $\Rightarrow$ (i). Let $x : X \rightarrowtail\!\!\!\twoheadrightarrow A$ be a repr.-preserving isomorphism. Make a global choice of slice objects with $(A{\downarrow}\mathsf{T}_{PA}, \delta^A_{\mathsf{T}_{PA}}) = (X, x)$. We have $Qx(\mathsf{T}_{QX}) = \chi_A(\mathsf{T}_{PA}) = \mathsf{T}_{QA}$, the first equality by definition of χ, the second equality by assumption.

(i) $\Rightarrow$ (iv). Let $f : B \rightarrowtail A$ be a discrete fibration. Then each $f{\downarrow}\mathsf{Y}$ is a repr.-preserving isomorphism, whence by assumption each $Q(f{\downarrow}\mathsf{Y})$ is repr. preserving, whence each $!_{Q(f{\downarrow}\mathsf{Y})}$ is an identity, whence so is each $\chi f(\mathsf{Y}) = Q\delta^A_{Pf(\mathsf{Y})}(!_{Q(f{\downarrow}\mathsf{Y})})$.

(iv) $\Rightarrow$ (v) follows from our introductory remark.

(v) $\Rightarrow$ (vi) is tautological.

(vi) ⇒ (i). Denote the transformation assumed to exist by $\tilde{\chi}$. Let $f : B \twoheadrightarrow A$ be a repr.-preserving isomorphism. In particular f is a discrete fibration, and so $Qf{\cdot}\tilde{\chi}_B = \tilde{\chi}_A{\cdot}Pf$ by assumption. The factors Pf, $\tilde{\chi}_A$ and $\tilde{\chi}_B$ are repr. preserving, the last two by assumption, and hence so is Qf.

As for the uniqueness claim, we inspect the construction in the proof of lemma 1.6: if both ϕ and ψ have the properties stated in (vi), then both $(\phi\delta^A_{\mathsf{X}})^{-1}$ and $\psi\delta^A_{\mathsf{X}}$ are identity natural transformations, and both $!_{\phi_{A\downarrow\mathsf{X}}}$ and $!^{\psi_{A\downarrow\mathsf{X}}}$ are identity morphisms in $Q(A{\downarrow}\mathsf{X})$; and so m (defined by (10)) is an identity modification. ■

The part of the proof that follows the right-hand cycle of (11) has two analogues, left for the reader to spell out, that yield the following results.

1.8. Proposition. *For the construction in the proof of lemma 1.5, the following four conditions are equivalent.*

(i) *Q preserves weak repr. preservation.*
(iv) *χ is weakly natural (on all of $\mathscr{A}$) for all global choices of slice objects.*
(v) *χ is weakly natural (on all of $\mathscr{A}$) for some global choice of slice objects.*
(vi) *There is some weakly repr.-preserving weakly natural transformation $P \longrightarrow Q$.* ■

1.9. Proposition. *For the construction in the proof of lemma 1.5, the following four conditions are equivalent.*

(i) *Q preserves (strict) repr. preservation.*
(iv) *χ is natural for all global choices of slice objects.*
(v) *χ is repr. preserving and natural for some global choice of slice objects.*
(vi) *There is some repr.-preserving natural transformation $P \longrightarrow Q$.*

Moreover, the transformation of (vi) *is unique.* ■

The latter proposition is of particular importance, as it allows us to swiftly complete the proof of theorem 1.4. If the functor Q is a slicing-site structure inducing the same factorization system on $\mathscr{A}$, then in particular it preserves repr. preservation. So there then is a unique repr.-preserving natural transformation $P \longrightarrow Q$. ■

We close this section with a summary of its main results.

1.10. Theorem. *On any category the repr.-preserving-isomorphism classes of slicing-site structures are in a specific one-to-one correspondence with the well-powering left-semistrict right-semireplete factorization systems. Hence, on an initially small category the repr.-preserving-isomorphism classes of slicing-site structures are in a one-to-one correspondence with all left-semistrict right-semireplete factorization systems.* ■

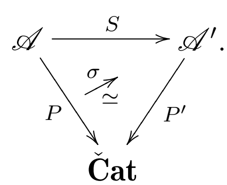
The word 'specific' is intended not just to refer to the construction given, but also to convey independence of the choices made in its course. Thus the question of naturality is evoked; it will be implicitly dealt with in the first half of the following section. The answer to it is predictably positive, but first it has to be given a definite meaning. That is, the two structure sets, one containing the classes of appropriate P, the other containing the appropriate $(\mathcal{L}, \mathcal{R})$, have to be interpreted as the respective values of two functors at the given category $\mathscr{A}$ as object of the domain. The morphisms of this domain will be the Grothendieck fibrations: they admit the pulling back of all slicing-site structures by precomposition and of all left-semistrict right-semireplete factorization systems in the manner of proposition 2.1; and they are the only functors to do the former.

2. Higher cells

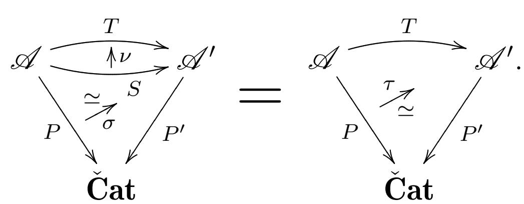
The typical kind

We have seen how a slicing-site structure can be given as a left-semistrict right-semireplete factorization system. The article [9] introduces not only slicing sites as individual entities, but also the structure governing the totality of all slicing sites, which is a 2-category. In this section we are going to see how the higher cells can be given directly in terms of the factorization systems.

We start by reproducing the definitions. Let $(\mathscr{A}, P)$ and $(\mathscr{A}', P')$ be two slicing sites. A 1-cell from $(\mathscr{A}, P)$ to $(\mathscr{A}', P')$ is a functor $S : \mathscr{A} \longrightarrow \mathscr{A}'$ that preserves the cartesianness of the slice-object projections in its domain, together with a repr.-preserving isomorphism $\sigma : P \overset{\simeq}{\Longrightarrow} P'S$:



Note that by proposition 1.9 (with $Q = P'S$), σ is determined by S. Now let (S, σ) and (T, τ) be two 1-cells from $(\mathscr{A}, P)$ to $(\mathscr{A}', P')$. A 2-cell from (S, σ) to (T, τ) is a natural transformation $\nu : S \longrightarrow T$ such that $P'\nu \cdot \sigma = \tau$:



(12)

Since σ and τ are repr. preserving, so is ν. Conversely, again by proposition 1.9 (this time with $Q = P'T$), any repr.-preserving natural transformation $\nu : S \longrightarrow T$ satisfies (12). The task for 2-cells is thus already completed; on the 1-cells we have to dwell a bit longer.

Fix a 1-cell $(S, \sigma) : (\mathscr{A}, P) \longrightarrow (\mathscr{A}', P')$; we are going to derive further properties of S.

By theorem 1.4 (uniqueness), the factorization system on $\mathscr{A}$ is determined by the one on $\mathscr{A}'$ and the functor S. How can we describe it directly in these terms?

Regarding just leftness we have the following. The presence of the repr.-preserving isomorphism σ implies that S preserves and reflects repr. preservation (the former by proposition 1.9, if you like): a morphism in $\mathscr{A}$ is repr. preserving precisely if so is its image under S in $\mathscr{A}'$.

Regarding rightness we could now use the right semirepleteness of $\mathscr{A}$; instead we seek a statement as similar as possible to the one just obtained. First note that the presence of the isomorphism σ also implies that S preserves and reflects the property of lying above a discrete fibration in $\check{\mathbf{C}}\mathbf{at}$. This leaves us with the question of cartesianness. On the one hand, we have seen that in a slicing site every discrete fibration plays the role of a slice-object projection, the cartesianness of which we are demanding to be preserved. Thus S respects discrete fibrations. (By corollary 1.2 it amounts to the same to say that S respects slice-object projections. But we can say even more: owing to its preserving repr., the natural transformation σ conveys the object "at which the slicing takes place". Explicitly, if (X, x) is a slice object of objects over X in A, then (SX, Sx) is a slice object of objects over $\sigma_A(\mathsf{X})$ in SA. This is a key feature of 1-cells.) On the other hand, it is well known that the class of those functors with respect to which a given morphism is cartesian has the "right" closure properties of section 1, in particular $(\mathrm{F2})_\mathrm{r}$ and $(\mathrm{F3})_\mathrm{r}$. Put precisely, in the situation $\mathscr{A} \xrightarrow{S} \mathscr{A}' \xrightarrow{S'} \mathscr{A}''$ a morphism in $\mathscr{A}$ whose image under S is cartesian with respect to S' is itself cartesian with respect to S precisely if it is cartesian with respect to $S'S$ (or any functor isomorphic to $S'S$). Applied to our situation (take $\mathscr{A}'' = \check{\mathbf{C}}\mathbf{at}$ and $S' = P'$) this yields that a morphism in $\mathscr{A}$ whose image under S is a discrete fibration is one itself precisely if it is cartesian *with respect to S*. Taking into account the preservation result, we conclude that a morphism in $\mathscr{A}$ is a discrete fibration precisely if it is cartesian with respect to S above a discrete fibration in $\mathscr{A}'$.

From the cited closure properties it also follows that S admits cartesian up-to-repr.-preserving-isomorphism liftings of discrete fibrations: for $A \in \mathscr{A}$ and $f' : B' \rightarrowtail SA \in \mathscr{A}'$, a cartesian up-to-repr.-preserving-isomorphism lifting of $P'f'$ along P is also a cartesian up-to-repr.-preserving-isomorphism lifting of f' along S.

It will turn out that the factorization-related properties we have discovered characterize the functors we are interested in. In order to make this statement precise, we introduce some terminology. Along the way we obtain a related result.

Let $\mathscr{A}$ and $\mathscr{A}'$ be two categories, the latter equipped with an (arbitrary) factorization system. By a *semifibration* $\mathscr{A} \longrightarrow \mathscr{A}'$ we shall mean a functor that admits cartesian up-to-both-sided-isomorphism liftings of right morphisms. For instance, a Grothendieck fibration to $\mathscr{A}'$ is precisely a semifibration with respect to $(\mathscr{A}'^0, \mathscr{A}')$, and a Street fibration to $\mathscr{A}'$ is precisely a semifibration with respect to $(\mathscr{A}'^\times, \mathscr{A}')$. In fact, a Grothendieck fibration is a semifibration with respect to any factorization system on its range, and a Street fibration is a semifibration with respect to any replete factorization system on its range.

2.1. Proposition. *Let $\mathscr{A}'$ be equipped with a factorization system $(\mathcal{L}', \mathcal{R}')$, and let $S : \mathscr{A} \longrightarrow \mathscr{A}'$ be a semifibration. Take $\mathcal{L}$ to be the class of morphisms above those in $\mathcal{L}'$, and take $\mathcal{R}$ to be the class of morphisms cartesian above those in $\mathcal{R}'$. $(\mathcal{L}, \mathcal{R})$ is a factorization system on $\mathscr{A}$. If $(\mathcal{L}', \mathcal{R}')$ is left semireplete, right semireplete, left semistrict or well-powering, then, respectively, so is $(\mathcal{L}, \mathcal{R})$.*

As far as replete factorization systems without additional properties are concerned, the result is part of the category-theoretic "folklore".

PROOF. With the exception of (F4), all properties for a pair of morphism classes highlighted in section 1 carry over individually from $(\mathcal{L}', \mathcal{R}')$ to $(\mathcal{L}, \mathcal{R})$, even without any assumption on the functor S. I can safely leave the verification of these implications to the reader. Nevertheless, as a sign of good will I consider explicitly the less familiar property $(\mathrm{F7})_\mathrm{l}$. Suppose $r{\cdot}l_0 = r{\cdot}l_1$ in $\mathscr{A}$ with l_0, $l_1 \in \mathcal{L}$ and $r \in \mathcal{R}$. Then $Sr{\cdot}Sl_0 = Sr{\cdot}Sl_1$ in $\mathscr{A}'$, where Sl_0, $Sl_1 \in \mathcal{L}'$ and $Sr \in \mathcal{R}'$. So $Sl_0 = Sl_1$. By cartesianness of r it follows that $l_0 = l_1$.

Next we consider (F4). Let $g : C \longrightarrow A$ be a morphism in $\mathscr{A}$. Its image Sg in $\mathscr{A}'$ has a factorization $SC \xrightarrow{l'} B' \xrightarrow{r'} SA$. Since S is a semifibration, r' has a cartesian up-to-both-sided-isomorphism lifting for A; that is, there are a cartesian $r : B \longrightarrow A$ and a $\beta' : SB \rightarrowtail\!\!\!\twoheadrightarrow B'$ such that $r'{\cdot}\beta' = Sr$. Cartesianness of r further yields an $l : C \longrightarrow B$ such that $r{\cdot}l = g$ and $Sl = \beta'^{-1}{\cdot}l'$. Since $r'{\cdot}\beta' \in \mathcal{R}'$ and $\beta'^{-1}{\cdot}l' \in \mathcal{L}'$ we further conclude that $r \in \mathcal{R}$ and $l \in \mathcal{L}$.

Well-poweringness also carries over without any assumption on S: the reader will easily construct a certain (repr.-preserving) functor $\sigma_A : PA \longrightarrow P'SA$ and show it to be injective on objects (and full) and faithful. (Under the assumption stated it is in fact invertible and could be obtained methodically via the 'everything small' interpretation of what would supersede the following proposition in section 3. That proposition itself covers the case of left semistrictness; right semirepleteness can be assumed without loss of generality.) ∎

The factorization system on $\mathscr{A}$ described here will be referred to as the one *induced by* S.

Proposition 2.1 gives rise to an alternative proof of theorem 1.3, founded on having already established the factorization system on $\check{\mathbf{C}}\mathbf{at}$ itself: slicing-site structures are precisely the semifibrations to $\check{\mathbf{C}}\mathbf{at}$, and the associated factorization system of the theorem is the induced one.

We can now present our result in a concise form.

2.2. PROPOSITION. *A functor between two slicing sites gives rise to a* 1*-cell as defined above if and only if it is a semifibration inducing the factorization system of its domain.*

PROOF. Continue using the above notation, where now S is an arbitrary functor $\mathscr{A} \longrightarrow \mathscr{A}'$. We have seen that the condition is necessary. As for its sufficiency, note that it implies (consider the preceding remark) that $P'S$ is a slicing-site structure inducing the same factorization system on $\mathscr{A}$ as does P. Thus, by theorem 1.4, there is a repr.-preserving isomorphism $P \simeq P'S$. ∎

Altogether we have obtained the following embellishment for theorem 1.10.

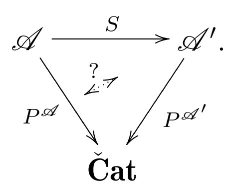
2.3. THEOREM. *There is a specific equivalence over* CAT *between the following two* 2*-categories: that of slicing sites and higher cells as defined above; that of categories with* *well-powering left-semistrict right-semireplete factorization systems, semifibrations inducing the factorization systems of their domains, and left natural transformations.* ∎

The all-inclusive kind

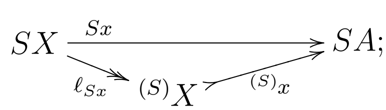
There will be little contention to the claim that the above definitions of higher cells of slicing sites are “correct” according to category-theoretic principles. Nevertheless, in [10] I shall have occasion to consider more general kinds of cells. Here is the salient example. Let $\mathscr{A}$ be an arbitrary slicing site. Take $\mathscr{A}'$ to be the same category equipped with the factorization system $(\mathscr{A}^0, \mathscr{A})$, and take S to be the identity functor $\mathscr{A} \longrightarrow \mathscr{A}'$. This functor is of interest as it corresponds to the canonical embedding of the category with slicing associated with $\mathscr{A}$ in a presheaf category. Yet it is a 1-cell in the above sense only if $\mathscr{A}$ as well carries the factorization system $(\mathscr{A}^0, \mathscr{A})$. In general it does not even have the seemingly desirable property of preserving both-sidedness. The following paragraphs deliver some thoughts on a class of cells that I hope contains all of the ones we shall ever encounter.

In this subsection we take ‘slicing site’ to mean: category equipped with a well-powering left-semistrict right-semireplete factorization system. For a slicing site $\mathscr{A}$ in this sense, we have the actual slicing-site structure (in the sense conveyed elsewhere) constructed towards theorem 1.4. Here we denote it by $P^{\mathscr{A}}$.

Let $\mathscr{A}$ and $\mathscr{A}'$ be two slicing sites. A 1-cell $\mathscr{A} \longrightarrow \mathscr{A}'$ should be an arbitrary functor S together with something that fills the 2-dimensional gap in

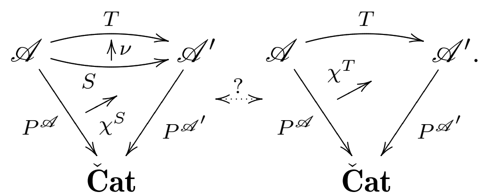


But there is an outstanding such thing, namely the laxly natural transformation $P^{\mathscr{A}} \longrightarrow P^{\mathscr{A}'}S$, weakly left and weakly natural on right morphisms, constructed in the proof of lemma 1.5. Here we denote it by χ^S. In fact, the construction depends on a global choice of slice objects for $(\mathscr{A}, P^{\mathscr{A}})$; but this we take to be the global choice of right factors for $\mathscr{A}$ already made in the construction of $P^{\mathscr{A}}$. Let me indicate the action of χ^S: for an object A of $\mathscr{A}$ and an object $[X, x]$ of $P^{\mathscr{A}}A$, take the preferred factorization of Sx, as displayed in the diagram

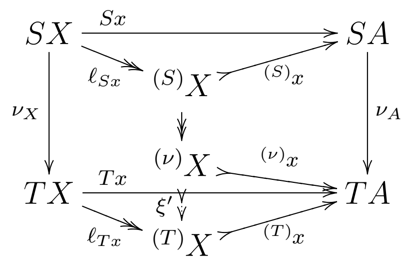


we have $\chi^S_A([X, x]) = [{}^{(S)}X, {}^{(S)}x]$. Note that replacing ‘$Q$’ and ‘$\chi$’ with ‘$S$’ and ‘$\chi^S$’ in propositions 1.7, 1.8 and 1.9 leaves the statements valid.

Now let S and T be two functors $\mathscr{A} \longrightarrow \mathscr{A}'$. Most generally, a 2-cell $S \longrightarrow T$ should be an arbitrary natural transformation ν together with something that fills the 3-dimensional gap in



But, as before, there is an outstanding such thing. Since ν is natural, the left-hand side inherits weak naturality on right morphisms from χ^S, and so by lemma 1.6 there is a unique modification $P^{\mathscr{A}'}\nu{\cdot}\chi^S \longrightarrow \chi^T$. We denote it by m^ν. It acts as follows: for an object A of $\mathscr{A}$ and an object $[X, x]$ of $P^{\mathscr{A}}A$, take the unique morphism ξ' rendering the diagram

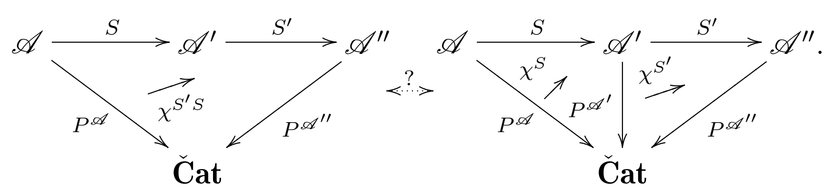


commutative; we have $\mathsf{m}^\nu_A([X, x]) = [\xi'] : [{}^{(\nu)}X, {}^{(\nu)}x] \longrightarrow [{}^{(T)}X, {}^{(T)}x]$. From this we can draw the following two conclusions:

- m^ν *is invertible if and only if* ν *is weakly left;*
- m^ν *is an identity if and only if* ν *is left.*

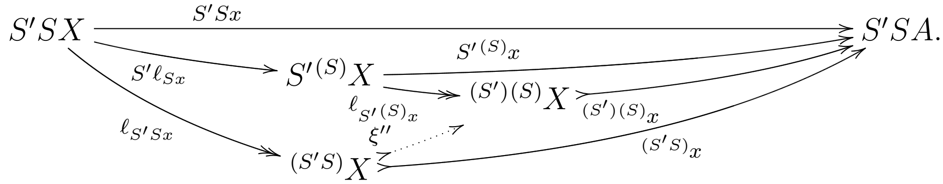
The former can also be derived more directly from the symmetrization of lemma 1.6. (The same would be true for the latter and proposition 1.7, if we were assuming that T respect both-sided isomorphisms. Without this assumption we depend on our global choice of slice objects for $\mathscr{A}$ being the same in the constructions of χ^S and χ^T.)

How do the assignments $P^{()}$, $\chi^{()}$ and $\mathsf{m}^{()}$ together behave towards composition? Let $\mathscr{A} \xrightarrow{S} \mathscr{A}' \xrightarrow{S'} \mathscr{A}''$ be a composable pair of functors between slicing sites. We want to compare the transformation associated with the composite with the composite of the associated transformations:



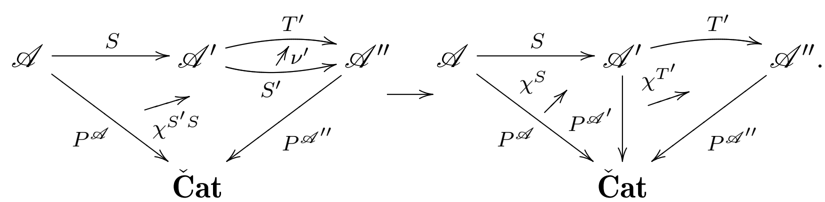
Again we can apply lemma 1.6: the right-hand side inherits weak leftness from χ^S and $\chi^{S'}$, and so there is a unique modification $\chi^{S'S} \longrightarrow \chi^{S'}S{\cdot}\chi^S$. We denote it by $\mathsf{c}^{S,S'}$. It acts as follows: $\mathsf{c}^{S,S'}_A$ maps an object $[X, x]$ of $P^{\mathscr{A}}A$ to the morphism $[\xi''] : [{}^{(S'S)}X, {}^{(S'S)}x] \longrightarrow$

$[{}^{(S')(S)}X, {}^{(S')(S)}x]$ that arises as can be gathered from the diagram



(13)

We have thus obtained a nullarily strict op-lax functor $\mathscr{A} \longmapsto (\mathscr{A}, P^{\mathscr{A}})$ (etc.) over CAT between two (by themselves rather insignificant) 2-categories. In the domain an object is a slicing site, a 1-cell is a functor, and a 2-cell is a natural transformation. (As a mere 2-category over CAT the domain is equivalent to CAT itself.) In one possible range an object is a category $\mathscr{A}$ together with a functor $P : \mathscr{A} \longrightarrow \check{\mathbf{C}}\mathbf{at}$, a 1-cell $(\mathscr{A}, P) \longrightarrow (\mathscr{A}', P')$ is a functor $S : \mathscr{A} \longrightarrow \mathscr{A}'$ together with a laxly natural transformation $\sigma : P \longrightarrow P'S$, and a 2-cell $(S, \sigma) \longrightarrow (T, \tau)$ is a natural transformation $\nu : S \longrightarrow T$ together with a modification $\mathsf{n} : \sigma \longrightarrow P'\nu{\cdot}\tau$. The various equalities making up the op-lax functoriality of $\mathscr{A} \longmapsto (\mathscr{A}, P^{\mathscr{A}})$ all are instances of the uniqueness part of lemma 1.6. By way of example, let us look at naturality in the (diagrammatically) right argument of 0-composition. We have to verify the equation $\mathsf{m}^{\nu'}S{\cdot}\chi^S \circ P^{\mathscr{A}''}\nu'S{\cdot}\mathsf{c}^{S,S'} = \mathsf{c}^{S,T'} \circ \mathsf{m}^{\nu'S}$ of modifications



The source is weakly natural on right morphisms, while the target is weakly left.

If we want $\mathscr{A} \longmapsto (\mathscr{A}, P^{\mathscr{A}})$ to be weak, so that we can obtain a bicategorical equivalence extending the 2-categorical one of theorem 2.3, we have to be more selective with regard to 1-cells. There are two principal options: either we demand preservation of weak leftness, or we demand preservation of rightness. This is made apparent by the following two propositions.

2.4. PROPOSITION. *Let $S' : \mathscr{A}' \longrightarrow \mathscr{A}''$ be a functor between slicing sites. The following three conditions are equivalent.*

(i) *S' respects weakly left morphisms.*

(ii) *$\mathsf{c}^{S,S'}$ is invertible for all slicing sites $\mathscr{A}$ and all functors $S : \mathscr{A} \longrightarrow \mathscr{A}'$.*

(iii) *$\mathsf{c}^{S,S'}$ is invertible for all slicing sites $\mathscr{A}$ and all functors $S : \mathscr{A} \longrightarrow \mathscr{A}'$ respecting weakly left morphisms.*

PROOF. (ii) ⇒ (iii) is tautological. If S' respects weakly left morphisms, then by proposition 1.8 $\chi^{S'}S$ is weakly natural; hence (i) ⇒ (ii) follows from the symmetrization of lemma 1.6. Now for (iii) ⇒ (i). We are assuming that in (13) ξ'' is invertible whenever S preserves weak leftness. Take $\mathscr{A}$ to be $\langle \bullet \overset{x}{\rightarrowtail} \bullet \rangle$, the free category on a single morphism, declared right and denoted by x. Given a weakly left morphism f' in $\mathscr{A}'$, take S to be the functor with $Sx = f'$. Then the right factor ${}^{(S)}x$ is invertible; hence so is the image $S'^{(S)}x$; hence so is the right factor ${}^{(S')(S)}x$; hence so is the composite ${}^{(S')(S)}x{\cdot}\xi'' = {}^{(S'S)}x$; hence $S'f' = S'Sx = {}^{(S'S)}x{\cdot}\ell_{S'Sx}$ is a weakly left morphism. ∎

2.5. Proposition. *Let $S : \mathscr{A} \longrightarrow \mathscr{A}'$ be a functor between slicing sites. Of the following three conditions, each except the last implies the next. For initially small $\mathscr{A}'$ they are equivalent.*

(i) *S respects right morphisms.*
(ii) *$\mathsf{c}^{S,S'}$ is invertible for all slicing sites $\mathscr{A}''$ and all functors $S' : \mathscr{A}' \longrightarrow \mathscr{A}''$.*
(iii) *$\mathsf{c}^{S,S'}$ is invertible for all slicing sites $\mathscr{A}''$ and all functors $S' : \mathscr{A}' \longrightarrow \mathscr{A}''$ respecting right morphisms.*

Proof. (ii) ⇒ (iii) is tautological. If S respects right morphisms, then $\chi^{S'}S$ is weakly natural on right morphisms; hence (i) ⇒ (ii) follows from the symmetrization of lemma 1.6. Now for (iii) ⇒ (i). We are assuming that $\mathscr{A}'$ is initially small and that in (13) ξ'' is invertible whenever S' preserves rightness. Take $\mathscr{A}''$ to be the category $\mathscr{A}'$ together with the factorization system $(\mathscr{A}'^{0}, \mathscr{A}')$, and take S' to be the identity functor. Let $f : B \rightarrowtail A$ be a right morphism in $\mathscr{A}$; we wish to show that Sf is a right morphism in $\mathscr{A}'$. Consider the preferred factorization $x{\cdot}\ell_f$ of f. Since the left factor ℓ_f, being right as well, is invertible, so is $S\ell_f$. Since $Sf = Sx{\cdot}S\ell_f$, this leaves us having to show that Sx is a right morphism. $\ell_{S'Sx}$ and $\ell_{S'^{(S)}x}$ are identities, whence $\ell_{Sx} = S'\ell_{Sx} = \xi''$ is invertible, whence $Sx = {}^{(S)}x{\cdot}\ell_{Sx}$ is right indeed. ∎

I have not managed to prove any of the backward implications under a sensible weaker assumption, such as for $\mathscr{A}'$ to be locally small. In general (ii) ⇒ (i) is false, as shown by the following counterexample. There is a category $\mathscr{D}$ (not locally small) with a morphism i that itself is not invertible, but whose image under any functor to **Set** is. Hence its image under any functor to any locally small category, in particular **Čat**, is invertible; hence its image under any functor to a slicing site is weakly left. Take $\mathscr{A}'$ to be $\mathscr{D}$ with the factorization system $(\mathscr{D}, \mathscr{D}^{\times})$. Take $\mathscr{A}$ to be $\langle \bullet \overset{x}{\rightarrowtail} \bullet \rangle$, and take S to be the functor sending x to i. Then clearly S does not preserve rightness, and the reader will check that in (13) ξ'' is always invertible.

If we want $\mathscr{A} \longmapsto (\mathscr{A}, P^{\mathscr{A}})$ to be strict, so that we can extend the 2-categorical equivalence of theorem 2.3 *as such*, there are again two principal options: either we demand preservation of *strict* leftness, or we demand preservation of rightness *and both-sidedness*. The propositions accompanying this assessment are somewhat less appealing.

2.6. Proposition. *Let $S' : \mathscr{A}' \longrightarrow \mathscr{A}''$ be a functor between slicing sites that respects both-sided morphisms. The following three conditions are equivalent.*

(i) *S' respects left morphisms.*
(ii) *$\mathsf{c}^{S,S'}$ is an identity for all slicing sites $\mathscr{A}$ and all functors $S : \mathscr{A} \longrightarrow \mathscr{A}'$.*
(iii) *$\mathsf{c}^{S,S'}$ is an identity for all slicing sites $\mathscr{A}$ and all functors $S : \mathscr{A} \longrightarrow \mathscr{A}'$ respecting left morphisms.*

(In fact, the premiss that S' preserve both-sidedness can be replaced with the weaker but bulkier condition that S' preserve the both-sidedness of just those morphisms chosen as the repr.s of the $P^{\mathscr{A}}A$. The latter holds true automatically provided the global choice of right factors for $\mathscr{A}'$ is normal; so by making normality obligatory we could obtain a perfect analogue of proposition 2.4.)

Proof. (ii) ⇒ (iii) is tautological. If S' respects left morphisms, then $S'\ell_{Sx}$ is left, whence so is ξ''; this shows (i) ⇒ (ii). The proof of (iii) ⇒ (i) is analogous to the one for proposition 2.4. ∎

2.7. Proposition. *Let $S : \mathscr{A} \longrightarrow \mathscr{A}'$ be a functor between slicing sites. Of the following three conditions, each except the last implies the next.*

(i) *S respects right morphisms.*
(ii) *$\mathsf{c}^{S,S'}$ is an identity for all slicing sites $\mathscr{A}''$ and all functors $S' : \mathscr{A}' \longrightarrow \mathscr{A}''$ respecting both-sided morphisms.*
(iii) *$\mathsf{c}^{S,S'}$ is an identity for all slicing sites $\mathscr{A}''$ and all functors $S' : \mathscr{A}' \longrightarrow \mathscr{A}''$ respecting right as well as both-sided morphisms.*

Proof. For a last time (ii) ⇒ (iii) is tautological. If S respects right morphisms, then Sx is right, whence ℓ_{Sx} is both-sided. If in addition S' respects both-sided morphisms, then $S'\ell_{Sx}$ is both-sided, hence left, whence so is ξ''. This shows (i) ⇒ (ii). ∎

A proof of a qualified (iii) ⇒ (i) analogous to the one for proposition 2.5 is not available. The obvious idea is to use $(\mathcal{L}'', \mathcal{R}'') = (\mathcal{L}' \cap \mathcal{R}', \mathscr{A}')$ as the factorization system on $\mathscr{A}''$, where $(\mathcal{L}', \mathcal{R}')$ is the given factorization system on $\mathscr{A}'$. It fails just because this $(\mathcal{L}'', \mathcal{R}'')$ in general is not left semistrict.

3. De-strictification

The generality of cells we considered in the latter half of section 2 rests on the 2^{nd} dimension of $\check{\mathbf{C}}\mathbf{at}$ — except for that of the objects. It suggests itself to try to extend the concept of a slicing site $(\mathscr{A}, P)$ in a way that allows for P to be a weak functor. Of course in doing so we should like to accordingly extend the little theory developed so far. And this can indeed be done, as we shall see in this last section.

Working out this extension is a largely straigtforward task, albeit a tedious one; the one difficulty is to find the proper set-up. A naïve approach, which will be considered in a preliminary stage, leads to a theory that is merely analogous to the prior one and has a defect of its own. The actual generalization will include the two as its extreme cases.

I am going to be terse on the formalities, leaving explicit proofs entirely to the reader. I shall, on the other hand, devote considerable attention to the rationale behind the definitions.

Weakness

The necessary adjustments in the shift from 'strict' to 'weak' can be subsumed by saying that we have to give up our 'up to isomorphism' view on the underlying categories in favour of an 'up to equivalence' one. Alas, we have actually been adopting an 'up to repr.-preserving isomorphism' view, which should get supplanted by an 'up to repr.-preserving equivalence' one, except that the latter notion is apparently ill-conceived, as a *weak* inverse of a repr.-preserving functor is not necessarily repr. preserving itself. Thus we see ourselves obliged to disregard repr.s altogether, putting all terminal objects on equal footing; said shift will as well concern repr. preservation. The place of **Čat** can be taken by the equivalent 2-category $\mathbf{Cat}_{(\times)}$.

This change of course squashes an important feature of our prior theory. If we nevertheless follow the present line of thought, rather than rectify it at its origin, we do so not least because the results it leads to have a special relevance: on the other side of the correspondence left semistrictness is traded for left semirepleteness, so that what arises are precisely factorization systems in the standard sense.

We proceed by adjusting the remaining ingredients in turn.

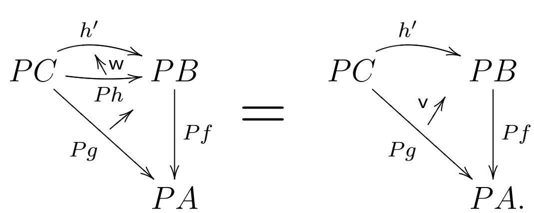
Cartesianness. For an ordinary functor $P : \mathscr{A} \longrightarrow \mathscr{K}$ of categories, a morphism $f : B \longrightarrow A$ in $\mathscr{A}$ is cartesian precisely if for each $g : C \longrightarrow A$ the induced map

$$(\mathscr{A}{\downarrow}A)((C,g),(B,f)) \longrightarrow (\mathscr{K}{\downarrow}PA)((PC,Pg),(PB,Pf)) \tag{14}$$

is bijective. Accordingly, for a strict/weak functor $P : \mathscr{A} \longrightarrow \mathscr{K}$ of 2-categories we call a 1-cell $f : B \longrightarrow A$ *strictly/weakly* cartesian if for each $g : C \longrightarrow A$ the induced functor (14), with '$\downarrow$' replaced by '$\Downarrow_0$'/'$\Downarrow_\times$', standing for the operation of forming the strict-/weak- slice 2-category, is strictly/weakly invertible (that is to say: is an isomorphism/equivalence). (There being no general implication from strict to weak cartesianness, we had better speak of, respectively, 2-*categorical* and *bicategorical* cartesianness, except that this terminology later yields an even bigger mouthful.)

Let us make this condition explicit, restricting ourselves to the pertinent case of a 1-category $\mathscr{A}$. Denote by $\mathscr{K}^{\circ\alpha}$, where α is the symbol '0'/'$\times$', the sub-2-category of identity/invertible 2-cells in $\mathscr{K}$. The strict/weak functor P has "composition preservators" with components $P^{(0)}{}_A : P1_A \longrightarrow 1_{PA}$ $(A \in \mathscr{A})$ and $P^{(2)}(h,f) : P(f{\cdot}h) \longrightarrow Pf{\cdot}Ph$ $(C \xrightarrow{h} B \xrightarrow{f} A \in \mathscr{A})$ belonging to $\mathscr{K}^{\circ\alpha}$. A morphism $f : B \longrightarrow A$ in $\mathscr{A}$ is strictly/weakly cartesian precisely if for each object C of $\mathscr{A}$ the following two conditions are satisfied.

(C1)$_\alpha$ For any morphism $g : C \longrightarrow B$ in $\mathscr{A}$, any 1-cell $h' : PC \longrightarrow PB$ in $\mathscr{K}$ and any 2-cell $\mathsf{v} : Pg \longrightarrow Pf{\cdot}h'$ in $\mathscr{K}^{\circ\alpha}$ there are, uniquely, a morphism $h : C \longrightarrow B$ in $\mathscr{A}$ with $f{\cdot}h = g$ and a 2-cell $\mathsf{w} : Ph \longrightarrow h'$ in $\mathscr{K}^{\circ\alpha}$ with $Pf{\cdot}\mathsf{w} \circ P^{(2)}(h,f) = \mathsf{v}$:



$(\mathrm{C2})_\alpha$ For any two 1-cells h'_0 and h'_1: $PC \longrightarrow PB$ and any 2-cell $\mathsf{w} : h'_0 \longrightarrow h'_1$ in $\mathscr{K}$, if $Pf{\cdot}\mathsf{w}$ is in $\mathscr{K}^{\circ\alpha}$, then so is w itself.

Note that, firstly, $(\mathrm{C1})_\alpha$ does not involve $\mathscr{K}$ as a whole, just its sub-2-category $\mathscr{K}^{\circ\alpha}$, and, secondly, $(\mathrm{C2})_\alpha$ does not involve f itself, just its image Pf. Condition $(\mathrm{C1})_0$ amounts to f being cartesian with respect to P as an ordinary functor (to $\mathscr{K}^{\circ 0}$, the 1-category underlying $\mathscr{K}$).

In the case we have been studying both these 2-dimensional concepts can stand in for the old 1-dimensional one. In order to make a general statement, we need the concept of a discrete fibration in an arbitrary 2-category, which is inherited via covariant representation: a 1-cell $f' : B' \longrightarrow A'$ is a discrete fibration in $\mathscr{K}$ provided each functor $\mathscr{K}(C', f') : \mathscr{K}(C', B') \longrightarrow \mathscr{K}(C', A')$ $(C' \in \mathscr{K})$ is an ordinary discrete fibration. *If $\mathscr{K}$ is a 2-category, $\mathscr{A}$ a 1-category, $P : \mathscr{A} \longrightarrow \mathscr{K}$ a strict functor, and f a morphism in $\mathscr{A}$ sent by P to a discrete fibration in $\mathscr{K}$, then the following three conditions on f are equivalent: weak cartesianness; strict cartesianness; cartesianness with respect to P as an ordinary functor.* It is interesting to note still a forth equivalent condition, namely the conjunction of $(\mathrm{C1})_1$ and $(\mathrm{C2})_1$, with $\mathscr{K}^{\circ 1}$ denoting $\mathscr{K}$ itself. Let me call it *op-lax* cartesianness, as its evident generalization to arbitrary op-lax functors P is sensible. (Thus the selection of directions for the invertible cells in the foregoing statements is exposed as deliberate.) There does not, however, appear to be a sensible generalization to arbitrary 2-categories $\mathscr{A}$.

Remark. Strict cartesianness was considered in [5], under the name '2-cartesian'. Weak cartesianness appears to be a new concept.

Skewness of liftings. For a functor $P : \mathscr{A} \longrightarrow \mathscr{K}$ of categories, an up-to-isomorphism lifting of $f' : B' \longrightarrow PA \in \mathscr{K}$ for $A \in \mathscr{A}$ is precisely an object (B, f) of $\mathscr{A}{\downarrow}A$ together with an isomorphism $\beta' : (P{\downarrow}A)(B, f) \xrightarrow{\simeq} (B', f')$ in $\mathscr{K}{\downarrow}PA$. Accordingly, for a weak functor $P : \mathscr{A} \longrightarrow \mathscr{K}$ of 2-categories we call an *up-to-equivalence* lifting of f' for A an object (B, f) of $\mathscr{A}{\Downarrow_\times}A$ together with an equivalence $(\beta', \mathsf{u}) : (P{\Downarrow_\times}A)(B, f) \xrightarrow{\approx} (B', f')$ in $\mathscr{K}{\Downarrow_\times}PA$, that is, an equivalence $\beta' : PB \xrightarrow{\approx} B'$ in $\mathscr{K}$ together with an invertible 2-cell $\mathsf{u} : Pf \xrightarrow{\simeq} f'{\cdot}\beta'$.

Rightness of underlying functors. In accordance with the weakening of the lifting requirements, the right-functor class has to be closed with respect to precomposition with equivalences and locally saturated with respect to isomorphism. Applied to the class of discrete (Grothendieck) fibrations the former operation forestalls the latter, as in fact all desirable closures, yielding the class of *essentially* discrete (Street) fibrations, that is, of those functors $f : B \longrightarrow A$ for which all the induced functors $f{\downarrow}\mathsf{Y} : B{\downarrow}\mathsf{Y} \longrightarrow A{\downarrow}f(\mathsf{Y})$ are equivalences. We note that above an essentially discrete fibration, weak and op-lax cartesianness with respect to a weak functor on a 1-category are equivalent.

The main definition reads as follows. A *weak slicing site* is a category together with a weak functor to $\mathbf{Cat}_{(\times)}$ admitting weakly cartesian up-to-equivalence liftings of slice-category projections. A morphism there is called repr. preserving if it lies above a terminality-preserving functor, and it is called a discrete fibration if it is weakly cartesian above an essentially discrete fibration. (We keep the names of the prior theory for simplicity.)

3.1. Theorem. *(Analogous to theorem 1.3.) For a weak slicing site, the class of repr.-preserving morphisms and the class of discrete fibrations form a replete factorization system.*

One part of the proof strays from the path of analogy, namely the verification of (F5). We had best start by establishing the relevant bicategorical orthogonality in $\mathbf{Cat}_{(\times)}$: for any terminality-preserving functor $l : D \longrightarrow C$ and any essentially discrete fibration $r : B \longrightarrow A$, the commutative square

$$\begin{array}{ccc} \mathscr{A}(C,B) & \xrightarrow{\mathscr{A}(C,r)} & \mathscr{A}(C,A) \\ {\scriptstyle \mathscr{A}(l,B)}\downarrow & & \downarrow{\scriptstyle \mathscr{A}(l,A)} \\ \mathscr{A}(D,B) & \xrightarrow[\mathscr{A}(D,r)]{} & \mathscr{A}(D,A) \end{array} \tag{15}$$

in **Cat**, where $\mathscr{A} := \mathbf{Cat}_{(\times)}$, is a bicategorical pullback; that is, the canonical functor $g \longmapsto (g{\cdot}l, r{\cdot}g, 1_{r\cdot g\cdot l})$ from the category of all functors $g : C \longrightarrow B$ to the category of all (h, f, i) consisting of two functors $h : D \longrightarrow B$ and $f : C \longrightarrow A$ and an invertible natural transformation $\mathsf{i} : f{\cdot}l \overset{\simeq}{\Longrightarrow} r{\cdot}h : D \longrightarrow A$, is an equivalence. (Taking $D = \mathbf{1}$ hence yields an analogue of a preliminary version of lemma 1.1.) Terminality-preserving functors and essentially discrete fibration in fact form a (replete) bicategorical factorization system (the definition of this concept can be found in section 1 of [3] with some detail and in section 5 of [12] with full generality) on $\mathbf{Cat}_{(\times)}$. This is induced by one on the whole of **Cat**, formed by final functors and essentially discrete fibrations. To conclude the proof we could provide and use a bicategorical generalization of the folklore version of proposition 2.1.

3.2. Theorem. *(Analogous to theorem 1.4.) Any well-powering replete factorization system $(\mathcal{L}, \mathcal{R})$ is induced in the manner of theorem 3.1 by a weak-slicing-site structure, unique up to a (weakly invertible) weakly natural transformation, unique up to a unique (invertible) modification.*

To obtain an existence proof analogous to that of theorem 1.3 we have to take picks for the left halves in what to call the preferred factorizations. Then, in the argument surrounding diagram (9) the preferred factorization $x{\cdot}m$ (say) of $(f{\cdot}h){\cdot}z$ comes with no guarantee that $m = l{\cdot}n$; all we can say is that there is a unique both-sided automorphism ξ of X such that $\xi{\cdot}m = l{\cdot}n$ and $x{\cdot}\xi = x$. By taking $P^{(2)}(h, f)([Z, z]) := [\xi]$ we obtain a weak functor P with the desired properties. This P in fact has an additional property, not adhering to the 'up to equivalence' paradigm, namely that each category PA is skeletal.

A fairly evident alternative construction offers a perhaps more interesting additional property. We take each object image PA to be the entire slice category $\mathcal{R}{\downarrow}A$, deferring the use of choice to the definition of the morphism images: for each $f : B \longrightarrow A$ and each $(Y, y) \in \mathcal{R}{\downarrow}B$ we pick a factorization $Y \overset{l}{\twoheadrightarrow} X \overset{x}{\rightarrowtail} A$ of $f{\cdot}y$ and put $Pf([Y, y]) := [X, x]$.

Here we have to make the stronger assumption that the right-morphism subcategory $\mathcal{R}$ is initially small. In the case that each object of $\mathscr{A}$ has only small-many automorphisms we can do so without loss of generality: first we replace $\mathscr{A}$ with an equivalent category whose isomorphism classes are small. By picking l to be the identity whenever f is right we can achieve that P has the strictness to make $\mathcal{R}$ a (strict) slicing site.

Uniqueness is dealt with by analogues of lemmata 1.5 and 1.6. (We also have an analogue of proposition 1.8, while propositions 1.7 and 1.9 in this respect suffer from references to strictness.)

3.3. Theorem. *(Analogous to theorem 1.10.) On any category the equivalence classes of weak-slicing-site structures are in a specific one-to-one correspondence with the well-powering replete factorization systems. Hence, on an initially small category the equivalence classes of weak-slicing-site structures are in a one-to-one correspondence with all replete factorization systems.*

T-weakness

To see what is needed for the general theory we had best approach the issue from the other side. It suggests itself to consider an arbitrary right-semireplete factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathscr{A}$. We get a weak-slicing-site structure P by taking PA to be $\mathcal{R}{\downarrow}A$ and so on, as mentioned above. But how will we be able to distinguish the structures resulting from different factorization systems whose (replete) right-morphism classes agree? The left-morphism classes can be included in one another, so that the choice of factorizations made for the smaller one works for the larger one as well. The answer is that we have to keep track of the traces of leftness, namely those morphisms in the $\mathcal{R}{\downarrow}A$ stemming from both-sided morphisms in $\mathscr{A}$. They are to form the "strata" of the PA in what follows.

To start with, we replace **Cat** with **Str Cat**, the category made up as follows. An object of **Str Cat** is a (small) *stratified* category, that is, a category together with a distinguished all-object subgroupoid, to be called the *stratum*. The (iso-)morphisms in the stratum will be called *T-isic* or the T-(iso-)morphisms. (The 'T' was the symbol for repr.s and is hence to allude to repr. preservation. Often I think of 'T-isomorphism' as a shorthand for 'true isomorphism'; unfortunately this phrasing of 'T-' is less elucidating in most of the other compounds.) The connected components of the stratum will be called the T-connected components. A morphism in **Str Cat** is a stratified functor, that is, a functor that preserves T-isicity, thus inducing a functor, also to be called the stratum, of the strata of domain and range.

We have the functor **Str Cat** $\longrightarrow$ **Cat** forgetting strata, which we denote by ∂. Conversely, we have two functors **Cat** $\longrightarrow$ **Str Cat**, one imagining strata consisting of identities only, one imagining strata consisting of all isomorphisms. The two are, respectively, a left adjoint and a right adjoint of ∂. We can use either to embed **Cat** in **Str Cat**, eventually making clear how the work of, respectively, section 1 and the previous subsection fit into the new context.

A morphism in **Str Cat** is cartesian with respect to ∂ precisely if it not just preserves, but also reflects T-isicity among invertible morphisms. ∂ admits unique cartesian liftings of all **Cat**-morphisms. We use this fact for the case of slice-category projections to introduce slicing in **Str Cat**. Explicitly, given a stratified category A and an object $\mathsf{X} \in A$, we stratify the category $A{\downarrow}\mathsf{X}$ with those morphisms stemming from the stratum of A; thus the functor δ^A_{X} becomes ∂-cartesianly stratified. (In fact, $(\mathbf{Str\,Cat}, \partial)$ is yet another example of a category with slicing.)

Str Cat is cartesian closed, thus self-enriched. All limits are created by ∂ in the evident manner. The exponential object A^B has objects the stratified functors $B \longrightarrow A$ and morphisms the natural transformations between these; the T-isomorphisms of A^B are those natural transformations all of whose components are T-isomorphisms of A. A **Str Cat**-enriched category is precisely a (locally small) locally stratified 2-category, that is, a 2-category together with a distinguished all-1-cell sub-2-category consisting of hom-groupoids. In what follows we could more generally consider locally stratified bicategories, which are defined analogously.

In a locally stratified 2-category we have the concept of a *T-equivalence*: a morphism that is T-weakly invertible, or, equivalently, either half of an adjunction whose unit and op-unit are T-isic. In **Str Cat** a T-equivalence is an ordinary equivalence with a stratum that is an equivalence as well, or, equivalently, a functor that is full and faithful, preserves and reflects T-isicity, and is surjective on T-connected components.

Now we replace $\check{\mathbf{Cat}}$ or $\mathbf{Cat}_{(\times)}$ with $\mathbf{Str\,Cat}_{(\square)}$: an object of $\mathbf{Str\,Cat}_{(\square)}$ is a (small) stratified category that is (T-weakly) represented, in the sense that it comes equipped with a distinguished T-connected component of terminal objects, called its (T-weak) repr.; a morphism in $\mathbf{Str\,Cat}_{(\square)}$ is simply a stratified functor. It is evident what it will mean for a stratified functor between represented stratified categories to be (T-weakly) repr. preserving. The left adjoint and the right adjoint of ∂ induce embeddings of $\check{\mathbf{Cat}}$ and $\mathbf{Cat}_{(\times)}$, respectively. The category $\mathbf{Str\,Cat}_{(\square)}$ inherits from **Str Cat** the structure to make it a locally stratified 2-category.

We can now redo the first half of this section. Rather than looking at $\mathbf{Cat}_{(\times)}$-valued weak functors, we have to look at $\mathbf{Str\,Cat}_{(\square)}$-valued T-weak functors, that is, weak functors whose composition preservators are T-isomorphisms. And rather than looking at the underlying categories up to equivalence, we have to look at the underlying stratified categories up to T-equivalence. We happily note that a T-weak inverse of a repr.-preserving stratified functor is repr. preserving as well.

Associated with a T-weak locally stratified functor $P : \mathscr{A} \longrightarrow \mathscr{K}$ of locally stratified 2-categories are the concepts of T-weak (better: *locally-stratified-bicategorical*) cartesianness and of an up-to-T-equivalence lifting. A 1-cell $f : B \longrightarrow A \in \mathscr{A}$ is T-weakly cartesian precisely if the functors (14), with '$\downarrow$' replaced by the symbol '$\Downarrow_\square$', which is to stand for the operation of forming the T-weak-slice locally stratified 2-category, are T-equivalences. Or, in the case of a 1-category $\mathscr{A}$, if conditions $(\mathrm{C1})_\square$ and $(\mathrm{C2})_\square$ are satisfied, with $\mathscr{K}^{\circ\square}$ denoting the sub-2-category of T-isic 2-cells. An up-to-T-equivalence lifting of $f' : B' \longrightarrow PA \in \mathscr{K}$ for $A \in \mathscr{A}$ is precisely an object (B, f) of $\mathscr{A}\Downarrow_\square A$ together with a T-equivalence $(\beta', \mathsf{u}) : (P\Downarrow_\square A)(B, f) \xrightarrow{\approx} (B', f')$ in $\mathscr{K}\Downarrow_\square PA$.

What about functor rightness? The role of discrete fibrations is played by the discrete T-fibrations, that is, those stratified functors $f : B \longrightarrow A$ for which all the induced (stratified) functors $f{\downarrow}\mathsf{Y} : B{\downarrow}\mathsf{Y} \longrightarrow A{\downarrow}f(\mathsf{Y})$ have stratified inverses (or, equivalently, those discrete fibrations that preserve and reflect T-isicity; or, equivalently, those discrete fibrations having strata that are discrete fibrations as well). Above a discrete T-fibration all our concepts of cartesianness with respect to a strict functor on a 1-category are equivalent. Closing off appropriately we obtain the T-essentially discrete T-fibrations, that is, those stratified functors f for which all the $f{\downarrow}\mathsf{Y}$ are T-equivalences. Above a T-essentially discrete T-fibration the concepts of T-weak, weak and op-lax cartesianness with respect to a T-weak functor on a 1-category are equivalent.

Here is the new main definition. A *T-weak slicing site* is a category together with a T-weak functor to $\mathbf{Str\,Cat}_{(\square)}$ admitting T-weakly cartesian up-to-repr.-preserving-T-equivalence liftings of slice-stratified-category projections. A morphism there is called repr. preserving if it lies above a repr.-preserving stratified functor, and it is called a discrete fibration if it is T-weakly cartesian above a T-essentially discrete T-fibration.

3.4. Theorem. *(Superseding theorems 1.3 and 3.1.) For a T-weak slicing site, the class of repr.-preserving morphisms and the class of discrete fibrations form a right-semireplete factorization system.*

We can define a (not necessarily replete) locally-stratified-bicategorical factorization system on a locally stratified 2-category $\mathscr{A}$ to consist of two 1-cell classes $\mathcal{L}$ and $\mathcal{R}$, locally saturated with respect to T-isomorphism, that fulfil conditions (F1–4) as well as the appropriate generalization of condition (F5): for any $l : A \longrightarrow B$ in $\mathcal{L}$ and any $r : C \longrightarrow D$ in $\mathcal{R}$ the commutative square (15) of stratified categories and stratified functors is a locally-stratified-bicategorical pullback; that is, the canonical stratified functor $g \longmapsto (g{\cdot}l, r{\cdot}g, 1_{r\cdot g\cdot l})$ from $\mathscr{A}(C,B)$ to the stratified category of all (h, f, i) consisting of $h \in \mathscr{A}(D,B)$, $f \in \mathscr{A}(C,A)$, $\mathsf{i} : f{\cdot}l \longrightarrow r{\cdot}h \in \mathscr{A}^{\circ\square}(D,A)$, is a T-equivalence. We can further call such a factorization system left or right semireplete provided $\mathcal{L}$ or $\mathcal{R}$, respectively, contains all the T-equivalences of $\mathscr{A}$. This will put us in a position to proclaim that $\mathbf{Str\,Cat}_{(\square)}$ carries a right-semireplete factorization system whose right 1-cells are the T-essentially discrete T-fibrations and whose left 1-cells are the those stratified functors that are repr. preserving, and that $\mathbf{Str\,Cat}$ carries a replete factorization system whose right 1-cells are the T-essentially discrete T-fibrations and whose left 1-cells are the those stratified functors that (as mere functors) are final.

3.5. Theorem. *(Superseding theorems 1.4 and 3.2.) Any well-powering right-semireplete factorization system $(\mathcal{L}, \mathcal{R})$ is induced in the manner of theorem 3.4 by a T-weak-slicing-site structure, unique up to a repr.-preserving (T-weakly invertible) T-weakly natural transformation, unique up to a unique T-isic modification.*

As regards existence, the extra step in the proof was mentioned at the beginning of this subsection. As regards uniqueness, the proof machinery of section 1 can be adjusted in its entirety: not just the two higher-cell lemmata, but also all three addenda to lemma 1.5

(‘strict’ and unqualified ‘unique’ to be replaced with ‘T-weak’ and ‘unique up to a unique T-isomorphism’, respectively).

3.6. Theorem. *(Superseding theorems 1.10 and 3.3.) On any category the repr.-preserving-T-equivalence classes of T-weak-slicing-site structures are in a specific one-to-one correspondence with the well-powering right-semireplete factorization systems. Hence, on an initially small category the repr.-preserving-T-equivalence classes of T-weak-slicing-site structures are in a one-to-one correspondence with all right-semireplete factorization systems.*

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