Theory and Applications of Categories, Vol. 28, No. 29, 2013, pp. 1002–1021.

RELATIVE MAL'TSEV CATEGORIES

TOMAS EVERAERT, JULIA GOEDECKE, TAMAR JANELIDZE-GRAY
AND TIM VAN DER LINDEN

Abstract. We define relative regular Mal'tsev categories and give an overview of conditions which are equivalent to the relative Mal'tsev axiom. These include conditions on relations as well as conditions on simplicial objects. We also give various examples and counterexamples.

1. Introduction

In recent years, the third author T. Janelidze-Gray and others have been working on extending the framework of *relative homological algebra* in the sense of [8] and [7, 33] to non-additive categories: see [28, 29, 30, 16, 15, 19]. In parallel with the "absolute" developments in [26, 1], this work gave rise to the notions of *relative semi-abelian* [30], *relative homological* [28] and *relative regular* [16] categories. Lying in between the latter two, there is the concept of *relative regular Mal'tsev category* which was already studied in [29]—though not explicitly named there. The path taken in [29] is to follow [6] and give characterisations of the concept in terms of internal equivalence relations.

Independently, in their article [11], the other three authors of the present paper introduce a very closely related framework involving a condition which they call the *relative Mal'tsev axiom*. They need this condition in the axiomatic study of the notion of *higher-dimensional extension* [14, 10] and its relationship to simplicial resolutions. In particular, they were looking for conditions which "go up to higher degrees" of extension, meaning that if the condition is satisfied in a category $\mathcal{A}$ for a chosen class of extensions $\mathcal{E}$, then it also holds in the category $\mathsf{Ext}(\mathcal{A})$ of extensions in $\mathcal{A}$ for the induced class $\mathcal{E}^1$ of so-called *double extensions* in $\mathcal{A}$. As we explain in this paper, while this approach is very close to T. Janelidze-Gray's relative homological algebra, the two are fundamentally incompatible. Nevertheless, part of the theory developed in [11] fits the relative homological

The first author's research was supported by Fonds voor Wetenschappelijk Onderzoek (FWO-Vlaanderen). The second author's research was supported by the FNRS grant *Crédit aux chercheurs* 1.5.016.10F. The third author's research was supported by the University of South Africa postdoctoral fellowship. The fourth author works as *chercheur qualifié* for Fonds de la Recherche Scientifique–FNRS. His research was supported by Fundação para a Ciência e a Tecnologia (grant number SFRH/BPD/38797/2007) and by CMUC at the University of Coimbra.

Received by the editors 2013-02-07 and, in revised form, 2013-10-03.

Transmitted by Stephen Lack. Published on 2013-10-14.

2010 Mathematics Subject Classification: 18A20, 18E10, 18G25, 18G30, 20J.

Key words and phrases: higher extension; simplicial resolution; Mal'tsev condition; relative homological algebra; arithmetical category.

algebraic picture, for instance its Theorem 3.13 which relates the relative Mal'tsev axiom to a relative Kan property of simplicial objects. Indeed, A. Carboni, G. M. Kelly and M. C. Pedicchio showed in [6] that in a regular category $\mathcal{A}$ every simplicial object being Kan is equivalent to $\mathcal{A}$ being a Mal'tsev category.

This naturally leads to the present article on relative Mal'tsev categories, in which we study the relative Mal'tsev axiom from [11] in the context of relative regular categories [16]. In particular, we show that the relative Mal'tsev axiom is equivalent to every $\mathcal{E}$-simplicial object satisfying a relative Kan property. We also explore a wide selection of examples, covering areas ranging from homological algebra via categorical Galois theory to torsion theories and including compact groups and internal groupoids.

In Section 2 we introduce the axioms for extensions which we use in the rest of the paper. In Section 3 we define relative (regular) Mal'tsev categories and study some of their properties, in particular relating to Kan simplicial objects. In Section 4 we explain why the context of relative regular categories does not match the perspective of [11]. The final section of the text is devoted to giving examples and counterexamples of relative Mal'tsev categories.

2. Axioms for extensions

The axioms we work with in this paper come from two different sources: some come from the world of relative homological and relative semi-abelian categories in the sense of T. Janelidze-Gray [28, 29, 30], and others revolve around the concept of *higher-dimensional extension* [14, 9, 10, 11]. All these axioms depend on a particular class $\mathcal{E}$ of arrows in a category $\mathcal{A}$. The basic axioms $\mathcal{E}$ should satisfy are:

(E1) $\mathcal{E}$ contains all isomorphisms;

(E2) pullbacks of morphisms in $\mathcal{E}$ exist in $\mathcal{A}$ and are in $\mathcal{E}$;

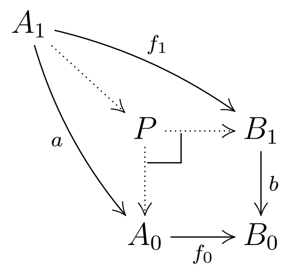
(E3) $\mathcal{E}$ is closed under composition.

2.1. Definition. If $\mathcal{E}$ satisfies (E1)–(E3), then a morphism in $\mathcal{E}$ is called an **extension**. We write $\mathsf{Ext}(\mathcal{A})$ for the full subcategory of the arrow category $\mathsf{Arr}(\mathcal{A})$ determined by the extensions.

Given $\mathcal{E}$, we now define the class $\mathcal{E}^1$ of **double extensions** in $\mathcal{A}$ as those morphisms $(f_1, f_0)\colon a \to b$

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & B_1 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A_0 & \xrightarrow[f_0]{} & B_0 \end{array}$$

in $\mathsf{Arr}(\mathcal{A})$ for which all arrows in the induced diagram



are in $\mathcal{E}$.

A useful point here is that $(\mathsf{Ext}(\mathcal{A}), \mathcal{E}^1)$ "inherits" the axioms (E1)–(E3) from $(\mathcal{A}, \mathcal{E})$, which allows us to iterate the definition to obtain also e.g. $(\mathsf{Ext}^2\mathcal{A}, \mathcal{E}^2)$.

2.2. Proposition. *Let $\mathcal{E}$ be a class of morphisms in a category $\mathcal{A}$. If $(\mathcal{A}, \mathcal{E})$ satisfies* (E1)–(E3)*, then so does* $(\mathsf{Ext}(\mathcal{A}), \mathcal{E}^1)$.

Proof. The proof of [14, Proposition 3.5] can be copied; see also [11, Proposition 1.6]. ■

The leading example for a class of extensions is the class of all regular epimorphisms in a regular category. Defining such classes of extensions axiomatically has two different benefits: on the one hand, it focuses on the essential properties needed for a given theory, and thus gives new examples, as we will see in the context of relative homological and relative semi-abelian categories [28, 30, 29] in Section 5. From a different viewpoint, it also allows the treatment of *higher* extensions and extensions at the same time, without needing to remember which "level" is needed at any given moment—see, for instance, [11, Proposition 3.11]. A collection of examples of such classes of extensions can be found at the end of this paper in Section 5, covering a wide range of areas. There are also examples in [11].

When the pair $(\mathcal{A}, \mathcal{E})$ satisfies additional axioms apart from (E1)–(E3) as defined above, more connections can be drawn to simplicial objects and in particular to a relative Kan property of simplicial objects. The axioms for a class of extensions $\mathcal{E}$ in a category $\mathcal{A}$ we shall use in this paper are:

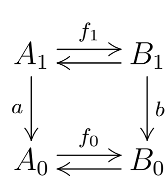
(E1) $\mathcal{E}$ contains all isomorphisms;

(E2) pullbacks of morphisms in $\mathcal{E}$ exist in $\mathcal{A}$ and are in $\mathcal{E}$;

(E3) $\mathcal{E}$ is closed under composition;

(E4⁻) if $f \in \mathcal{E}$ and $g \circ f \in \mathcal{E}$ then $g \in \mathcal{E}$;

(E5⁻) the **$\mathcal{E}$-Mal'tsev axiom**: any split epimorphism of extensions



in $\mathcal{A}$ with f_1 and f_0 in $\mathcal{E}$ is a double extension.

Some examples in a pointed category $\mathcal{A}$ also satisfy the stronger axiom

(E5$^+$) given a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(a) & \longrightarrow & A_1 & \xrightarrow{a} & A_0 & \longrightarrow & 0 \\ & & {\scriptstyle k}\downarrow & & {\scriptstyle f}\downarrow & & \| & & \\ 0 & \longrightarrow & \mathrm{Ker}(b) & \longrightarrow & B & \xrightarrow[b]{} & A_0 & \longrightarrow & 0 \end{array}$$

in $\mathcal{A}$ with short exact rows and a and b in $\mathcal{E}$, if $k \in \mathcal{E}$ then also $f \in \mathcal{E}$.

Note that, in a pointed category, Axiom (E2) ensures the existence of kernels of extensions.

These axioms are satisfied, for example, by all **relative homological categories** as defined in [28]. These are pairs $(\mathcal{A}, \mathcal{E})$, where $\mathcal{A}$ is a pointed category with finite limits and cokernels, and $\mathcal{E}$ is a class of normal epimorphisms in $\mathcal{A}$ satisfying axioms (E1)–(E3), (E4$^-$) and (E5$^+$), as well as the axiom

(F) if a morphism f in $\mathcal{A}$ factors as $f = e \circ m$ with m a monomorphism and $e \in \mathcal{E}$, then it also factors (essentially uniquely) as $f = m' \circ e'$ with m' a monomorphism and $e' \in \mathcal{E}$.

This axiom (F) allows us, amongst other things, to prove that certain split epimorphisms are in fact extensions.

2.3. Lemma. *If $\mathcal{A}$ has finite products, $\mathcal{E}$ is a class of epimorphisms in $\mathcal{A}$ and $(\mathcal{A}, \mathcal{E})$ satisfies* (E1)–(E3) *and* (F)*, then given any split epimorphism of extensions*

$$\begin{array}{ccccc} A_1 \times_{A_0} A_1 & \rightrightarrows & A_1 & \xrightarrow{a} & A_0 \\ {\scriptstyle r}\downarrow\uparrow & & {\scriptstyle f_1}\downarrow\uparrow & & {\scriptstyle f_0}\downarrow\uparrow \\ B_1 \times_{B_0} B_1 & \rightrightarrows & B_1 & \xrightarrow[b]{} & B_0 \end{array}$$

with f_1 and f_0 in $\mathcal{E}$, taking kernel pairs of a and b gives an extension r.

Proof. Consider a diagram as above and the composite morphism

$$A_1 \times_{A_0} A_1 \xrightarrow{\langle \pi_0, \pi_1 \rangle} A_1 \times A_1 \xrightarrow{f_1 \times f_1} B_1 \times B_1.$$

The product $f_1 \times f_1$ is an extension by (E2) and (E3), and $\langle \pi_0, \pi_1 \rangle$ is a monomorphism. Hence by (F) the morphism $(f_1 \times f_1) \circ \langle \pi_0, \pi_1 \rangle$ admits a factorisation $\langle r_0, r_1 \rangle \circ e$, where (R, r_0, r_1) is a relation on B_1 and e is in $\mathcal{E}$. Since e is an epimorphism by assumption, we have $b \circ r_0 = b \circ r_1$, and R is contained in $B_1 \times_{B_0} B_1$. Now r being a split epimorphism implies that $R = B_1 \times_{B_0} B_1$. ∎

2.4. Remark. We can now justify why Axiom (E5$^+$) is "stronger" than (E5$^-$): Suppose (E1)–(E3) and (F) hold and $\mathcal{E}$ consists of normal epimorphisms. Consider a split epimorphism of extensions as in (E5$^-$). Take the kernels of a and b to obtain a split epimorphism of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(a) & \xrightarrow{\ker(a)} & A_1 & \xrightarrow{a} & A_0 & \longrightarrow & 0 \\ & & {\scriptstyle k}\downarrow\uparrow & & {\scriptstyle f_1}\downarrow\uparrow & & {\scriptstyle f_0}\downarrow\uparrow & & \\ 0 & \longrightarrow & \mathrm{Ker}(b) & \xrightarrow[\ker(b)]{} & B_1 & \xrightarrow[b]{} & B_0 & \longrightarrow & 0 \end{array}$$

Now a similar, but in fact easier, argument as in the proof of Lemma 2.3 shows that k is an element of $\mathcal{E}$. So (E5$^+$) implies that the right hand square is a double extension.

Axiom (E5$^-$) is connected to some other conditions on double extensions. To prove these connections, we first need:

2.5. Lemma. *Let $(\mathcal{A},\mathcal{E})$ satisfy* (E1)–(E4$^-$) *and* (F), *and consider a diagram*

$$\begin{array}{ccccc} A_1\times_{A_0}A_1 & \overset{\pi_0}{\underset{\pi_1}{\rightrightarrows}} & A_1 & \xrightarrow{a} & A_0 \\ {\scriptstyle r}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow \\ B_1\times_{B_0}B_1 & \overset{\pi_0'}{\underset{\pi_1'}{\rightrightarrows}} & B_1 & \xrightarrow[b]{} & B_0 \end{array}$$

with a, b, f_1 and f_0 in $\mathcal{E}$. Then either of the left hand squares is in $\mathcal{E}^1$ if and only if the right hand square is in $\mathcal{E}^1$.

Proof. See [11, Lemma 3.2]. ∎

2.6. Proposition. *Let $(\mathcal{A},\mathcal{E})$ satisfy* (E1)–(E3) *and* (E4$^-$). *Consider the following statements:*

(i) (E4$^-$) *holds for $\mathcal{E}^1$, that is, if $g\circ f\in\mathcal{E}^1$ and $f\in\mathcal{E}^1$ then $g\in\mathcal{E}^1$;*

(ii) *Axiom* (E5$^-$) *holds;*

(iii) *every split epimorphism of split epimorphisms with a, b, f_1 and f_0 in $\mathcal{E}$, i.e. every diagram*

$$\begin{array}{ccc} A_1 & \overset{\overline{f_1}}{\underset{f_1}{\leftrightarrows}} & B_1 \\ {\scriptstyle a}\downarrow\uparrow{\scriptstyle \overline{a}} & & {\scriptstyle b}\downarrow\uparrow{\scriptstyle \overline{b}} \\ A_0 & \overset{\overline{f_0}}{\underset{f_0}{\leftrightarrows}} & B_0, \end{array}$$

such that $f_0a=bf_1$, $\overline{f_0}b=a\overline{f_1}$, $\overline{b}f_0=f_1\overline{a}$, $\overline{a}\overline{f_0}=\overline{f_1}\overline{b}$ and $f_0\overline{f_0}=1_{B_0}$, $f_1\overline{f_1}=1_{B_1}$, $a\overline{a}=1_{A_0}$, $b\overline{b}=1_{B_0}$ and the four split epimorphisms are in $\mathcal{E}$, is a double extension;

(iv) *given a diagram*

$$\begin{array}{ccccc} A_1 \times_{B_1} A_1 & \rightrightarrows & A_1 & \xrightarrow{f_1} & B_1 \\ {\scriptstyle r}\downarrow & & {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A_0 \times_{B_0} A_0 & \rightrightarrows & A_0 & \xrightarrow[f_0]{} & B_0 \end{array}$$

in $\mathcal{A}$ *with* a, b, f_1 *and* f_0 *in* $\mathcal{E}$, *the arrow* r *is in* $\mathcal{E}$ *if and only if the right hand side square is in* $\mathcal{E}^1$.

Then (ii) $\Rightarrow$ (iv) $\Rightarrow$ (i) *and* (ii) $\Rightarrow$ (iii). *If* $(\mathcal{A}, \mathcal{E})$ *also satisfies* (F), *then* (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii), *resulting in the equivalence of* (ii), (iii) *and* (iv).

Proof. Clearly (iii) is a special case of (ii). In (iv), the right to left implication always holds by pullback-stability (E2) for $\mathcal{E}^1$. The other direction follows easily from (ii) and Lemma 2.5. The part (iv) $\Rightarrow$ (i) follows by translating between the double arrows $f\colon a \to b$, $g\colon b \to c$ and gf and the induced morphisms between the kernel pairs of a, b and c with (iv) and using (E4$^-$) on the latter.

Now, using (F), Lemma 2.3 immediately gives (iv) $\Rightarrow$ (ii). For (iii) $\Rightarrow$ (iv), consider a diagram as in (iv) and take kernel pairs upwards of the left hand square. Axiom (F), via Lemma 2.3 again, is needed to see that the resulting square is of the type given in (iii), so then using Lemma 2.5 twice gives the result. ■

It can be seen that (E1)–(E4$^-$) and (E5$^-$) "go up to higher dimensions together", meaning:

2.7. Proposition. *Let* $\mathcal{A}$ *be a category and* $\mathcal{E}$ *a class of arrows in* $\mathcal{A}$. *If* $(\mathcal{A}, \mathcal{E})$ *satisfies* (E1)–(E4$^-$) *and* (E5$^-$), *then* $(\mathsf{Ext}(\mathcal{A}), \mathcal{E}^1)$ *satisfies the same conditions.*

Proof. The axioms (E1)–(E3) were already treated in Proposition 2.2. Axiom (E4$^-$) goes up by (ii) $\Rightarrow$ (i) in Proposition 2.6. For (E5$^-$) it suffices to notice that the proof of [11, Proposition 3.4] is still valid. ■

3. The relative Mal'tsev axiom and relations

Classically, Mal'tsev categories are defined using properties of relations. Therefore we now connect the relative Mal'tsev condition (E5$^-$) to the conditions on $\mathcal{E}$-relations studied in [30, 29]. For this, we use a context given in Condition 2.1 in [30], that is, we assume that $\mathcal{A}$ has finite products, $\mathcal{E}$ is a class of regular epimorphisms in $\mathcal{A}$ and $(\mathcal{A}, \mathcal{E})$ satisfies axioms (E1)–(E3), (E4$^-$) and (F). In [16] such a pair $(\mathcal{A}, \mathcal{E})$ is called a **relative regular category**. For a more detailed explanation see [30] and [16].

3.1. Definition. Given two objects A and B in $\mathcal{A}$, an **$\mathcal{E}$-relation from A to B** is a subobject of $A \times B$ such that for any representing monomorphism $\langle r_0, r_1\rangle\colon R \to A \times B$, the morphisms $r_0\colon R \to A$ and $r_1\colon R \to B$ are in $\mathcal{E}$.

Using the axioms given, such $\mathcal{E}$-relations can be composed and this composition is associative. The usual definitions and calculations of relations apply. This setting allows us to copy proofs and methods from [6] to the relative context. Many of these results were proved in [29, Theorem 2.3.6]; in particular, for a relative regular category $(\mathcal{A}, \mathcal{E})$, we have:

3.2. Proposition. *For any relative regular category $(\mathcal{A}, \mathcal{E})$, the following are equivalent:*

(i) *for equivalence $\mathcal{E}$-relations R and S on an object A in $\mathcal{A}$, the relation $SR\colon A \to A$ is an equivalence $\mathcal{E}$-relation;*

(ii) *any two equivalence $\mathcal{E}$-relations R and S on an object A in $\mathcal{A}$ permute: $SR = RS$;*

(iii) *any two $\mathcal{E}$-effective equivalence relations R and S (i.e., kernel pairs of extensions) on A in $\mathcal{A}$ permute;*

(iv) *every $\mathcal{E}$-relation is difunctional;*

(v) *every reflexive $\mathcal{E}$-relation is an equivalence $\mathcal{E}$-relation;*

(vi) *every reflexive $\mathcal{E}$-relation is symmetric;*

(vii) *every reflexive $\mathcal{E}$-relation is transitive.* ∎

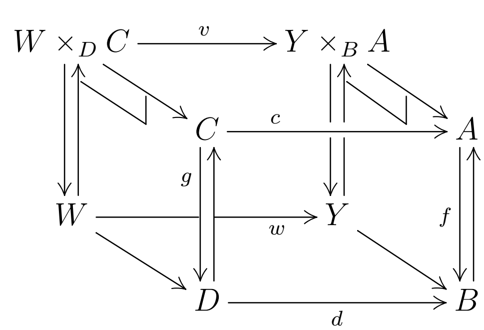
All these conditions are equivalent to our relative Mal'tsev axiom (E5$^-$), as M. Gran and D. Rodelo showed in their paper [19]. In fact, they also showed that (E5$^-$) is equivalent to several other conditions, including a condition on relations and a diagram lemma called the *Relative Cuboid Lemma*:

3.3. Theorem. [19] *If $(\mathcal{A}, \mathcal{E})$ is a relative regular category, then the following are equivalent:*

(i) *Axiom* (E5$^-$)*;*

(ii) *any two $\mathcal{E}$-effective equivalence relations R and S on A in $\mathcal{A}$ permute;*

(iii) *for any commutative cube*



in $\mathcal{A}$, where f and g are split epimorphisms in $\mathcal{E}$, c, d, and w are in $\mathcal{E}$, and the left and right squares are pullbacks, the induced morphism $v\colon W \times_D C \to Y \times_B A$ is an extension;

(iv) *the* Relative Split Cuboid Lemma *holds;*

(v) *the* Relative Upper Cuboid Lemma *holds.* ■

We are now ready to give the following

3.4. Definition. A relative regular category $(\mathcal{A}, \mathcal{E})$ is **relative Mal'tsev** if it satisfies any one of the conditions 3.2(i)–3.2(vii) or 3.3(i)–3.3(v) above.

Note that any relative regular Mal'tsev category is **relative Goursat** in the sense of [16]: for equivalence $\mathcal{E}$-relations R and S on an object A, the equality $RSR = SRS$ holds. Hence in any relative regular Mal'tsev category, also the *Relative* 3×3 *Lemma* is valid—see [32, 18, 16].

We are now finally approaching our main result about the relative Mal'tsev axiom: its characterisation in terms of the $\mathcal{E}$-Kan property for $\mathcal{E}$-simplicial objects.

3.5. Definition. Let $\mathbb{A}$ be a simplicial object and consider $n \geqslant 2$ and $0 \leqslant k \leqslant n$. The **object of** (n, k)**-horns in** $\mathbb{A}$ is an object $A(n, k)$ together with arrows $a_i\colon A(n, k) \to A_{n-1}$ for $i \in \{0, \ldots, n\}\backslash\{k\}$ satisfying

$$\partial_i \circ a_j = \partial_{j-1} \circ a_i \text{ for all } i < j \text{ with } i,\, j \neq k$$

which is universal with respect to this property. We also define $A(1, 0) = A(1, 1) = A_0$.

A simplicial object is **$\mathcal{E}$-Kan** when all $A(n, k)$ exist and all comparison morphisms $A_n \to A(n, k)$ are in $\mathcal{E}$. In particular, the comparison morphisms to the $(1, k)$-horns are $\partial_0\colon A_1 \to A(1, 0)$ and $\partial_1\colon A_1 \to A(1, 1)$.

For the proof, we will need a property of contractible $\mathcal{E}$-Kan simplicial objects:

3.6. Proposition. *In a relative regular category $(\mathcal{A}, \mathcal{E})$, an augmented $\mathcal{E}$-simplicial object $\mathbb{A}$ which is contractible and $\mathcal{E}$-Kan is always an $\mathcal{E}$-resolution: for all $n \geqslant -1$, the factorisation $A_{n+1} \to K_{n+1}\mathbb{A}$ to the simplicial kernel $K_{n+1}\mathbb{A}$ of $\partial_0, \ldots, \partial_n\colon A_n \to A_{n-1}$ (and $K_0\mathbb{A} = A_{-1}$) is in $\mathcal{E}$.*

Proof. As $\mathbb{A}$ is an $\mathcal{E}$-semi-simplicial object, in particular the morphism

$$\partial_0\colon A_0 \to A_{-1} = K_0\mathbb{A}$$

is in $\mathcal{E}$, so $\mathbb{A}$ is an $\mathcal{E}$-resolution at level 0.

Now let $\mathbb{A}$ be a resolution up to level n. We can assume inductively that the simplicial kernel $K_{n+1}\mathbb{A}$ exists (see [11, Lemma 3.8], which uses only axioms (E1)–(E3)). So in the diagram

$$
\begin{array}{ccccccc}
A_{n+2} & \longrightarrow & A(n+2,0) & \overset{a_1}{\underset{a_{n+2}}{\rightrightarrows}} \vdots & A_{n+1} & \overset{\partial_1}{\underset{\partial_{n+1}}{\rightrightarrows}} \vdots & A_n \\
{\scriptstyle \partial_0}\downarrow & & {\scriptstyle r}\downarrow\uparrow & & {\scriptstyle \partial_0}\downarrow\uparrow{\scriptstyle \sigma_{-1}} & & {\scriptstyle \partial_0}\downarrow\uparrow{\scriptstyle \sigma_{-1}} \\
A_{n+1} & \underset{\langle \partial_0,\dots,\partial_{n+1}\rangle}{\longrightarrow} & K_{n+1}\mathbb{A} & \overset{k_0}{\underset{k_{n+1}}{\rightrightarrows}} \vdots & A_n & \overset{\partial_0}{\underset{\partial_n}{\rightrightarrows}} \vdots & A_{n-1}
\end{array}
$$

we have to prove that $\langle \partial_0, \dots, \partial_{n+1}\rangle$ is an extension. Here $A(n+2,0)$ and $K_{n+1}\mathbb{A}$ are the simplicial kernels for the given morphisms. As $\mathbb{A}$ is $\mathcal{E}$-Kan, $A_{n+1} \to A(n+2,0)$ is an extension, and ∂_0 is an extension by assumption. So to be able to use (E3) and (E4$^-$) , it only remains to show that r is an extension. This is done as in the proof of Lemma 2.3. ■

3.7. Theorem. *Let $(\mathcal{A}, \mathcal{E})$ be a relative regular category such that $\mathcal{A}$ has simplicial kernels. Then $(\mathcal{A}, \mathcal{E})$ is relative Mal'tsev if and only if every $\mathcal{E}$-simplicial object in $\mathcal{A}$ is $\mathcal{E}$-Kan.*

Proof. For this proof we use (E5$^-$) out of the equivalent definitions defining a relative Mal'tsev category. The direction $\Rightarrow$ is proved by induction using symmetry properties of higher extensions, see [11, Proposition 3.11].

Conversely, when (E1)–(E4$^-$) and (F) hold and every $\mathcal{E}$-simplicial object is $\mathcal{E}$-Kan, we wish to show that every split epimorphism of split epimorphisms with all appropriate arrows in $\mathcal{E}$ is a double extension. This then implies (E5$^-$) by Proposition 2.6. We can first reduce the situation to a (truncated) contractible augmented $\mathcal{E}$-simplicial object

$$
A_1 \;\underset{\sigma_1}{\overset{\sigma_{-1}}{\begin{array}{c}\xrightarrow{\partial_0} \\ \xleftarrow{\sigma_0} \\ \xrightarrow{\partial_1}\end{array}}}\; A_0 \;\underset{\partial_0}{\overset{\sigma_{-1}}{\rightleftarrows}}\; A_{-1}. \qquad \textbf{(A)}
$$

Given a split epimorphism of split epimorphisms

$$
\begin{array}{ccc}
A & \overset{\overline{f}}{\underset{f}{\leftrightarrows}} & B \\
{\scriptstyle a}\downarrow\uparrow{\scriptstyle \overline{a}} & & {\scriptstyle b}\downarrow\uparrow{\scriptstyle \overline{b}} \\
A' & \overset{\overline{f'}}{\underset{f'}{\leftrightarrows}} & B'
\end{array}
$$

with a, b, f and f' in $\mathcal{E}$, we define $A_{-1} = B'$, $A_0 = A$, $\partial_0 = f' \circ a = b \circ f \colon A_0 \to A_{-1}$ and $\sigma_{-1} = \overline{a} \circ \overline{f'} = \overline{f} \circ \overline{b} \colon A_{-1} \to A_0$. The morphisms ∂_0 and $\partial_1 \colon A_1 \to A_0$ are defined by the pullback

$$\begin{array}{ccc} A_1 & \xrightarrow{\;p\;} & A_0 \\ {\scriptstyle \langle \partial_0, \partial_1 \rangle}\big\downarrow & & \big\downarrow{\scriptstyle \langle a, f \rangle} \\ A_0 \times_{A_{-1}} A_0 & \xrightarrow[a \times_{1_{B'}} f]{} & A' \times_{B'} B \end{array} \qquad (\mathbf{B})$$

where the morphism $a \times_{1_{B'}} f$ is an extension as the pullback of the double extensions $(f' \circ a, f') \colon a \to 1_{B'}$ and $(f' \circ a, b) \colon f \to 1_{B'}$. The morphisms σ_{-1}, $\sigma_0 \colon A_0 \to A_1$ are induced by

$$(a \times_{1_{B'}} f) \circ \langle 1_{A_0}, 1_{A_0} \rangle = \langle a, f \rangle \circ 1_{A_0}$$

and

$$(a \times_{1_{B'}} f) \circ \langle 1_{A_0}, \overline{a} \circ \overline{f'} \circ f' \circ a \rangle = \langle a, f \rangle \circ (\overline{a} \circ a)$$

respectively. We also need $\sigma_1 \colon A_0 \to A_1$ induced by

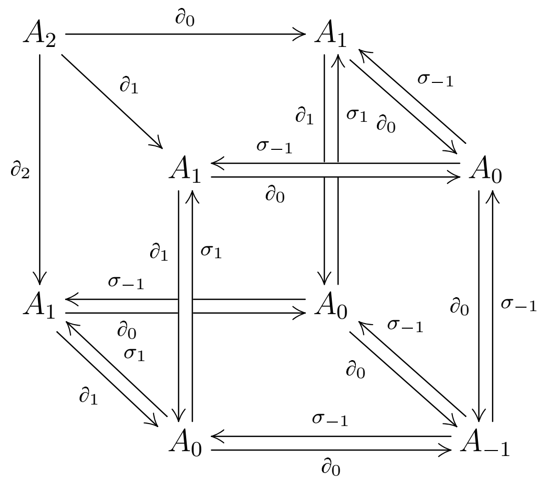
$$(a \times_{1_{B'}} f) \circ \langle \overline{f} \circ \overline{b} \circ b \circ f, 1_{A_0} \rangle = \langle a, f \rangle \circ (\overline{f} \circ f).$$

These morphisms then satisfy the simplicial identities; in particular, $\partial_1 \circ \sigma_1 = 1_{A_0}$ and $\partial_0 \circ \sigma_1 = \sigma_{-1} \circ \partial_0$. It remains to check that ∂_0 and ∂_1 are also extensions. We may decompose the diagram defining, say, ∂_0, as

$$\begin{array}{ccccc} A_1 & \xrightarrow{\;\overline{r}\;} & Q & \longrightarrow & A_0 \\ {\scriptstyle \langle \partial_0, \partial_1 \rangle}\big\downarrow & & {\scriptstyle \overline{\langle a, f \rangle}}\big\downarrow & & \big\downarrow{\scriptstyle \langle a, f \rangle} \\ A_0 \times_{A_{-1}} A_0 & \xrightarrow{\;r\;} & P & \longrightarrow & A' \times_{B'} B \\ {\scriptstyle \pi_0}\big\downarrow & & {\scriptstyle \overline{\pi_{A'}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{A'}} \\ A_0 & = & A_0 & \xrightarrow[a]{} & A'. \end{array}$$

The induced morphism r is an extension (since the bottom rectangle is a double extension), hence so is $\overline{r}$. The composite $\overline{\pi_{A'}} \circ \overline{\langle a, f \rangle}$ is also an extension, as a pullback of $a = \pi_{A'} \circ \langle a, f \rangle$. Hence $\partial_0 = \pi_0 \circ \langle \partial_0, \partial_1 \rangle$ is an extension by (E3). Similarly, so is ∂_1.

A truncated $\mathcal{E}$-simplicial object of the shape $(\mathbf{A})$ can be extended to a contractible augmented simplicial object $\mathbb{A}$ by constructing successive simplicial kernels. Using (F) we now show that such a simplicial object is actually an $\mathcal{E}$-simplicial object, so that it is $\mathcal{E}$-Kan by assumption. To see this, we write $(\mathbf{A})$ in the form of a cube, where A_2 is the induced simplicial kernel. The simplicial identities ensure that all possible squares in it commute.



The simplicial kernel property of A_2 makes this cube a limit diagram (see e.g. Theorem 2.17 in [11] for an explanation). Taking pullbacks in the front and back faces of the cube we obtain the induced square

$$\begin{array}{ccc} A_2 & \xrightarrow{\partial_1} & A_1 \\ {\scriptstyle\langle\partial_2,\partial_0\rangle}\downarrow & & \downarrow{\scriptstyle\langle\partial_1,\partial_0\rangle} \\ A_1 \times_{A_0} A_1 & \xrightarrow[\partial_1 \times_{\partial_0} \partial_0]{} & A_0 \times_{A_{-1}} A_0 \end{array}$$

which is also a pullback by the limit property of A_2. Using a similar argument as in the proof of Lemma 2.3, we see that the morphism $\partial_1 \times_{\partial_0} \partial_0$ is an extension. Hence $\partial_1 \colon A_2 \to A_1$ is also in $\mathcal{E}$. By symmetric arguments, so are ∂_0 and $\partial_2 \colon A_2 \to A_1$, making $\mathbb{A}$ an $\mathcal{E}$-simplicial object up to A_2.

For the induction step, remember that the universal property of A_n induces degeneracies/contractions σ_{-1} to $\sigma_n \colon A_{n-1} \to A_n$ satisfying the simplicial identities. Given a simplicial kernel such as A_{n+1} of $n+1$ given morphisms $\partial_0, \ldots, \partial_n \colon A_n \to A_{n-1}$ which themselves form a simplicial kernel, the $n+1$ first morphisms $\partial_0, \ldots, \partial_n \colon A_{n+1} \to A_n$ form a simplicial kernel of the morphisms $\partial_0, \ldots, \partial_{n-1} \colon A_n \to A_{n-1}$. Hence, by induction, all face maps of $\mathbb{A}$ are in $\mathcal{E}$. Therefore, by Proposition 3.6, $\mathbb{A}$ is an $\mathcal{E}$-resolution. In particular, the induced comparison morphism $\langle \partial_0, \partial_1 \rangle \colon A_1 \to A_0 \times_{A_{-1}} A_0$ in Diagram (**B**) is an extension. Using (E4$^-$) on Diagram (**B**), we conclude that the original split epimorphism of split epimorphisms is a double extension. ∎

4. On the axiom (F) and higher dimensions

As we mentioned in Section 2, one advantage of treating extensions in an axiomatic setting is to be able to treat higher dimensions more easily. Axiom (F) is of a slightly different flavour than the other axioms, and we now explain under which conditions, in the absolute case, Axiom (F) goes up to higher dimensions. Here $\mathcal{E}$ is the class of all regular epimorphisms in $\mathcal{A}$. We restrict to this absolute case in order to use results about arithmetical categories which are only written down in the absolute case; similar arguments will also work in the relative setting, but would take more work to write out in detail, and this absolute setting is enough to make our point.

4.1. REMARK. Note that a morphism $f = (f_1, f_0)\colon a \to b$ between extensions a and b is a monomorphism in $\mathsf{Ext}(\mathcal{A})$ if and only if f_1 is a monomorphism. In particular, there are no restrictions on f_0. When $\mathcal{A}$ is regular, pushouts of regular epimorphisms are exactly the regular epimorphisms in $\mathsf{Ext}(\mathcal{A})$.

4.2. PROPOSITION. *Let $\mathcal{A}$ be a regular category and $\mathcal{E}$ the class of all regular epimorphisms in $\mathcal{A}$. The following conditions are equivalent:*

(i) *$\mathcal{A}$ is exact Mal'tsev;*

(ii) *the pushout of an extension by an extension exists and is a double extension;*

(iii) $(\mathsf{Ext}(\mathcal{A}), \mathcal{E}^1)$ *satisfies* (F).

PROOF. The equivalence of (i) and (ii) was proved by A. Carboni, G. M. Kelly and M. C. Pedicchio in [6]. Assuming (ii), any morphism $f\colon a \to b$ in $\mathsf{Ext}(\mathcal{A})$ factors as a double extension followed by a monomorphism as follows.

$$\begin{array}{ccccc} A_1 & \xrightarrow{e} & I & \xrightarrow{m} & B_1 \\ {\scriptstyle a}\downarrow & \Rightarrow & \downarrow & \Rightarrow & \downarrow{\scriptstyle b} \\ A_0 & \longrightarrow & P & \longrightarrow & B_0 \end{array}$$

Here $f_1 = m \circ e$ is the regular epi-mono factorisation of f_1 and the left hand square is the pushout of e by a. Note that the former exists because $\mathcal{A}$ is regular and the latter by assumption. Hence, (ii) implies (iii). To see that (iii) implies (ii), consider extensions f and g and the morphism of extensions

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & \Rightarrow & \downarrow \\ C & \longrightarrow & 1 \end{array}$$

where 1 is the terminal object. This square can be factored as a monomorphism (in the category of extensions) followed by a double extension as follows.

$$\begin{array}{ccccc} A & = & A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & \Rightarrow & \downarrow & \Rightarrow & \downarrow \\ C & \longrightarrow & 1 & = & 1 \end{array}$$

The assumption implies that the square can also be factored as a double extension followed by a monomorphism.

$$\begin{array}{ccccc} A & \xrightarrow{e} & I & \xrightarrow{m} & B \\ {\scriptstyle g}\downarrow & \Rightarrow & \downarrow & \Rightarrow & \downarrow{\scriptstyle b} \\ C & \longrightarrow & I' & \longrightarrow & 1 \end{array}$$

But this means in particular that m is a monomorphism. Hence, it is an isomorphism, since it is also a regular epimorphism (as f is). It follows that the pushout of f by g exists (it is given by the left hand square) and is a double extension, as desired. ■

Let us now investigate under which circumstances (F) "goes up" to $(\mathsf{Ext}^2(\mathcal{A}), \mathcal{E}^2)$. Clearly, as soon as $(\mathsf{Ext}^2(\mathcal{A}), \mathcal{E}^2)$ satisfies (F), the same will be true for $(\mathsf{Ext}(\mathcal{A}), \mathcal{E}^1)$. Hence, by Proposition 4.2, a necessary condition for $(\mathsf{Ext}^2(\mathcal{A}), \mathcal{E}^2)$ to satisfy (F) is that $\mathcal{A}$ is exact Mal'tsev. Observe that, in this case, $\mathsf{Ext}(\mathcal{A})$ is regular: regular epimorphisms in $\mathsf{Ext}(\mathcal{A})$ are double extensions, which we know are pullback-stable. Hence, we can apply Proposition 4.2 to $\mathsf{Ext}(\mathcal{A})$ and find, in particular, that the pair $(\mathsf{Ext}^2(\mathcal{A}), \mathcal{E}^2)$ satisfies (F) if and only if $\mathsf{Ext}(\mathcal{A})$ is exact Mal'tsev.

Now, recall from [34] that an exact Mal'tsev category is **arithmetical** if every internal groupoid is an equivalence relation. Examples of arithmetical categories are the dual of the category of pointed sets, more generally, the dual of the category of pointed objects in any topos, and also the categories of von Neumann regular rings, Boolean rings and Heyting semi-lattices. It was proved in [3] that an exact Mal'tsev category is arithmetical if and only if the category $\mathsf{Equiv}(\mathcal{A})$ of internal equivalence relations in $\mathcal{A}$ is exact. In this case $\mathsf{Equiv}(\mathcal{A})$ is in fact again arithmetical and, in particular, exact Mal'tsev. Since, moreover, there is a category equivalence $\mathsf{Equiv}(\mathcal{A}) \simeq \mathsf{Ext}(\mathcal{A})$ because $\mathcal{A}$ is exact, we have:

4.3. Proposition. *Let $\mathcal{A}$ be an exact Mal'tsev category and $\mathcal{E}$ the class of all regular epimorphisms in $\mathcal{A}$. The following are equivalent:*

- *$\mathcal{A}$ is arithmetical;*
- *$\mathsf{Ext}(\mathcal{A})$ is arithmetical;*
- *$\mathsf{Ext}(\mathcal{A})$ is exact Mal'tsev;*
- *any pushout of a double extension by a double extension exists (in the category $\mathsf{Ext}(\mathcal{A})$) and is a three-fold extension;*
- *$(\mathsf{Ext}^2(\mathcal{A}), \mathcal{E}^2)$ satisfies (F).* ■

4.4. Remark. Note that Proposition 4.3 also implies that Axiom (F) is satisfied by $(\mathsf{Ext}^n(\mathcal{A}), \mathcal{E}^n)$ for *every* n as soon as the category $\mathcal{A}$ is arithmetical. Conversely, the category $\mathcal{A}$ is arithmetical as soon as there exists an $n \geqslant 2$ such that (F) holds for $(\mathsf{Ext}^n(\mathcal{A}), \mathcal{E}^n)$.

Since being arithmetical is a rather restrictive property for a (Mal'tsev) category to have, we can conclude this analysis by saying that Axiom (F) "hardly ever" goes up to $(\mathsf{Ext}^2(\mathcal{A}), \mathcal{E}^2)$ or higher.

This shows that, while Axiom (F) fits very well into the context of relations and relative homological and semi-abelian categories, it is not necessarily the best context for higher extensions. In the paper [11], three of the present authors treat the relative Mal'tsev axiom in a different context which does lend itself very well to the study of higher extensions. The axioms in that context are (E1)–(E3) as well as

(E4) if $g \circ f \in \mathcal{E}$ then $g \in \mathcal{E}$ (right cancellation);

(E5) the **$\mathcal{E}$-Mal'tsev axiom**: any split epimorphism of extensions

$$\begin{array}{ccc} A_1 & \overset{f_1}{\rightleftarrows} & B_1 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A_0 & \underset{}{\overset{f_0}{\rightleftarrows}} & B_0 \end{array}$$

in $\mathcal{A}$ is a double extension.

This right cancellation axiom is clearly a stronger version of the weak cancellation axiom (E4$^-$), and (E1) together with (E4) imply that all split epimorphisms are in $\mathcal{E}$. The precise connections are:

4.5. Proposition. *Let $(\mathcal{A}, \mathcal{E})$ satisfy* (E1)–(E4$^-$). *Then $\mathcal{E}$ contains all split epimorphisms if and only if* (E4) *holds.*

Proof. By (E1), one of the implications is obvious. To prove the other, let $g \circ f$ be in $\mathcal{E}$. Pulling back induces the following commutative diagram:

$$\begin{array}{ccccc} P & \xrightarrow{\overline{f}} & B \times_C B & \underset{\pi_1}{\rightleftarrows} & B \\ \uparrow\downarrow{\scriptstyle \overline{\pi_0}} & & \uparrow\downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C. \end{array}$$

The split epimorphism $\overline{\pi_0}$ is in $\mathcal{E}$ by assumption. Furthermore, the composite $\pi_1 \circ \overline{f}$ is in $\mathcal{E}$ by (E2). Now (E3) and (E4$^-$) imply that g is in $\mathcal{E}$. ∎

Clearly, when $\mathcal{E}$ contains all split epimorphisms, (E5$^-$) is equivalent to (E5). When $\mathcal{E}$ consists of normal epimorphisms, Axiom (E5$^+$) also implies (E5), thus making sense of our naming convention.

5. Examples

We end this article with several examples and counterexamples. Some of the examples satisfy the stronger axiom (E5$^+$), cf. [2, 9, 10, 28].

5.1. Example. [Relative homological categories] As mentioned in Section 2, relative homological and relative semi-abelian categories as defined in [28, 30] are relatively Mal'tsev, but generally they need not satisfy the stronger (E4) and (E5). An example of a relative semi-abelian category is a semi-abelian category $\mathcal{A}$ with $\mathcal{E}$ being the class of central extensions in the sense of Huq, closed under composition [29, Proposition 5.3.2]; see also Example 5.4. That is, any morphism in $\mathcal{E}$ is the composition of regular epimorphisms $f\colon A \to B$ with $[\mathrm{Ker}(f), A] = 0$, where $[\mathrm{Ker}(f), A]$ is the commutator of $\mathrm{Ker}(f)$ and A in the sense of Huq [21].

When $\mathcal{E}$ is a class of regular epimorphisms in a regular Mal'tsev category $\mathcal{A}$ satisfying (E1)–(E2), then it is easy to check that (E3), ($E4^-$) and ($E5^-$) hold as soon as the following **two out of three property** is satisfied: given a composite $g \circ f$ of regular epimorphisms $f\colon A \to B$ and $g\colon B \to C$, if any two of $g \circ f$, f and g lie in $\mathcal{E}$, then so does the third. We shall make use of this fact when considering the following two examples, which are given by categorical Galois theory [22, 23]. Note that this uses the regular Mal'tsev property to show that, in the square given in ($E5^-$), the comparison to the pullback is already a regular epimorphism, and then the two out of three property shows that it is in fact in $\mathcal{E}$.

5.2. Example. [Trivial extensions] Let $\mathcal{B}$ be a full and replete reflective subcategory of a regular Mal'tsev category $\mathcal{A}$. Write $H\colon \mathcal{B} \to \mathcal{A}$ for the inclusion functor and $I\colon \mathcal{A} \to \mathcal{B}$ for its left adjoint. Assume that HI preserves regular epimorphisms and I is *admissible* [23] with respect to regular epimorphisms. This means that I preserves all pullbacks of the form

$$\begin{array}{ccc} B \times_{HI(B)} H(X) & \longrightarrow & H(X) \\ \downarrow & & \downarrow{\scriptstyle H(\varphi)} \\ B & \underset{\eta_B}{\longrightarrow} & HI(B) \end{array} \qquad \textbf{(C)}$$

where $\varphi\colon X \to I(B)$ is a regular epimorphism. For instance, $\mathcal{B}$ could be a Birkhoff subcategory of $\mathcal{A}$ (a full reflective subcategory closed under subobjects and regular quotients) if $\mathcal{A}$ is also Barr-exact (see [24]).

Recall that a **trivial covering** or **trivial extension** (with respect to I) is a regular epimorphism f such that the commutative square induced by the unit $\eta\colon 1_{\mathcal{A}} \Rightarrow HI$

$$\begin{array}{ccc} A & \overset{\eta_A}{\longrightarrow} & HI(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle HI(f)} \\ B & \underset{\eta_B}{\longrightarrow} & HI(B) \end{array} \qquad \textbf{(D)}$$

is a pullback. With $\mathcal{E}$ the class of all trivial extensions, $(\mathcal{A}, \mathcal{E})$ satisfies conditions (E1)–($E4^-$) and ($E5^-$); see also [31]. (The stronger axiom (E4) need not hold as in general not every split epimorphism is a trivial extension: for instance, when $\mathcal{A}$ is pointed, a morphism $A \to 0$ is a trivial extension if and only if A is in $\mathcal{B}$.) Indeed, the validity of (E1) is clear while (E2) follows from the admissibility of I (see Proposition 2.4 in [25]). Hence, it suffices to prove the two out of three property, of which only one implication is not immediate. To see that $g\colon B \to C$ is a trivial extension as soon as $f\colon A \to B$ and $g\circ f$ are, it suffices to note that, since $HI(f)$ is a pullback-stable regular epimorphism, the change of base functor $(HI(f))^*\colon (\mathcal{A} \downarrow HI(B)) \to (\mathcal{A} \downarrow HI(A))$ is conservative [27].

When $\mathcal{A}$ is Barr-exact and $\mathcal{B}$ is a Birkhoff subcategory of $\mathcal{A}$, then $(\mathcal{A}, \mathcal{E})$ also satisfies (F). Indeed, condition (F) is easily inferred from the fact that in this case the square (**D**) is a pushout, hence a regular pushout (a double extension) for any regular epimorphism f [6, 24]. If moreover $\mathcal{A}$ is pointed with cokernels and $\mathcal{B}$ is protomodular, then $(\mathcal{A}, \mathcal{E})$ forms a relative homological category [31].

5.3. Example. [Torsion theories] Recall that $p\colon E \to B$ is an **effective descent morphism** if the change of base functor $p^*\colon (\mathcal{A} \downarrow B) \to (\mathcal{A} \downarrow E)$ is monadic. Let $\mathcal{A}$ be a homological category in which every regular epimorphism is effective for descent (for instance, $\mathcal{A}$ could be semi-abelian) and let $\mathcal{B}$ be a torsion-free subcategory of $\mathcal{A}$ (a full regular epi-reflective subcategory of $\mathcal{A}$ such that the associated radical $T\colon \mathcal{A} \to \mathcal{A}$ is idempotent, see [5]). Then the reflector $I\colon \mathcal{A} \to \mathcal{B}$ is semi-left exact: it preserves all pullbacks of the form (**C**), now for *all* morphisms $\varphi\colon X \to I(B)$. In particular, the previous example applies. Thus we find that the pair $(\mathcal{A}, \mathcal{E})$ satisfies conditions (E1)–(E4$^-$) and (E5$^-$), for $\mathcal{E}$ the class of all trivial extensions.

Let us now write $\mathcal{E}^*$ for the class of (regular epi)morphisms $f\colon A \to B$ that are "locally in $\mathcal{E}$", in the sense that there exists an effective descent morphism $p\colon E \to B$ in $\mathcal{A}$ such that the pullback $p^*(f)\colon E \times_B A \to E$ is in $\mathcal{E}$. The morphisms in $\mathcal{E}^*$ are usually called **coverings** or **central extensions**. While the pair $(\mathcal{A}, \mathcal{E}^*)$ satisfies conditions (E1) and (E2) because $(\mathcal{A}, \mathcal{E})$ does, $\mathcal{E}^*$ is in general not closed under composition. However, it was shown in [13] that $\mathcal{E}^*$ is composition-closed as soon as the reflector I is **protoadditive** [12, 13]: I preserves split short exact sequences. Let us briefly recall the argument. First of all, it was shown in [13] that the central extensions with respect to I (which we shall, from now on, assume to be protoadditive) are exactly those regular epimorphisms $f\colon A \to B$ whose kernel $\mathrm{Ker}(f)$ is in $\mathcal{B}$. Now, let $f\colon A \to B$ and $g\colon B \to C$ be regular epimorphisms. Then we have a short exact sequence

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow \mathrm{Ker}(g\circ f) \longrightarrow \mathrm{Ker}(g) \longrightarrow 0$$

and we see that $g\circ f$ is a central extension as soon as f and g are, since the torsion-free subcategory $\mathcal{B}$ is closed under extensions (which means that when $\mathrm{Ker}(f) \in \mathcal{B}$ and $\mathrm{Ker}(g) \in \mathcal{B}$ then $\mathrm{Ker}(g\circ f) \in \mathcal{B}$) [5]. Furthermore, since $\mathcal{B}$ is a (regular epi)-reflective subcategory of $\mathcal{A}$, $\mathcal{B}$ is closed under subobjects, and so f is a central extension as soon as $g\circ f$ is. If we assume that $\mathcal{B}$ is, moreover, closed under regular quotients (which is equivalent to $\mathcal{B}$ being a Birkhoff subcategory of $\mathcal{A}$) then g is a central extension as soon as $g\circ f$ is, and we may conclude that $\mathcal{E}^*$ satisfies the two out of three property. Once again using that $\mathcal{B}$ is closed under subjects in $\mathcal{A}$, it is easily verified that the pair $(\mathcal{A}, \mathcal{E}^*)$ also satisfies Axiom (F). (Note that the same two out of three property can be used to show that $(\mathcal{A}, \mathcal{E}^*)$ is, in fact, relatively homological.)

Examples of such an $\mathcal{A}$ and $\mathcal{B}$ are given, for instance, by taking $\mathcal{A}$ to be the category of compact Hausdorff groups and $\mathcal{B}$ the subcategory of profinite groups [13], or $\mathcal{A}$ to be the category of internal groupoids in a semi-abelian category and $\mathcal{B}$ the subcategory of discrete groupoids [12]. Since a reflector into an epi-reflective subcategory of an abelian category is necessarily (proto)additive, any cohereditary torsion theory (meaning that $\mathcal{B}$ is closed under quotients) in an abelian category $\mathcal{A}$ provides an example as well. However, there are no non-trivial examples in the categories of groups or of abelian groups, as follows from Proposition 5.5 in [35].

5.4. Example. [Composites of central extensions] We use the context of Example 5.2, assuming in addition that $\mathcal{A}$ is Barr-exact and $\mathcal{B}$ is a Birkhoff subcategory of $\mathcal{A}$. In this setting a regular epimorphism $f\colon A \to B$ is a central extension (with respect to I) if there exists a regular epimorphism $p\colon E \to B$ such that the pullback $p^*(f)\colon E \times_B A \to E$ of f along p is a trivial extension. We take $\mathcal{E}$ to be the class of composites of such central extensions. If now $\mathcal{A}$ is pointed and has cokernels and coproducts, and $\mathcal{B}$ is protomodular, then $(\mathcal{A}, \mathcal{E})$ forms a relative semi-abelian category [31]. When $\mathcal{B}$ is determined by the abelian objects in $\mathcal{A}$, we regain the example mentioned in 5.1: then the $\mathcal{B}$-central extensions in $\mathcal{A}$ are determined by the Smith commutator [4], while, via [20], extensions are Smith-central if and only if they are Huq-central as in Example 5.1.

5.5. Example. [Internal groupoids] Let the pair $(\mathcal{A}, \mathcal{E})$ satisfy axioms (E1)–(E4$^-$), (E5$^-$) and (F). Denote by $\mathsf{Gpd}_{\mathcal{E}}(\mathcal{A})$ the category of **internal $\mathcal{E}$-groupoids** in $\mathcal{A}$: groupoids G in $\mathcal{A}$ with the property that all split epimorphisms occurring in the diagram of G are in $\mathcal{E}$. Write $\overline{\mathcal{E}}$ for the class of degree-wise $\mathcal{E}$-extensions. Then $(\mathsf{Gpd}_{\mathcal{E}}(\mathcal{A}), \overline{\mathcal{E}})$ is relatively Mal'tsev. Indeed, to see that axioms (E2) and (E5$^-$) are satisfied, observe that pullbacks along morphisms in $\overline{\mathcal{E}}$ are degree-wise pullbacks in $\mathcal{A}$. For Axiom (F) note that products are computed degree-wise as well, and that $\mathsf{Gpd}_{\mathcal{E}}(\mathcal{A})$ is closed in $\mathsf{RG}_{\mathcal{E}}(\mathcal{A})$—the category of "reflexive $\mathcal{E}$-graphs" in $\mathcal{A}$—under "$\overline{\mathcal{E}}$-quotients", as a consequence of the relative Mal'tsev condition for $(\mathcal{A}, \mathcal{E})$. See [17] for the absolute case.

5.6. Example. [Regular pullback squares] This is an example of a pair $(\mathcal{A}, \mathcal{E})$ which satisfies (E1)–(E4$^-$) and (E5$^-$), but where not every split epimorphism is an extension, nor does (F) hold. We take $\mathcal{A}$ to be the category $\mathsf{Ext}(\mathsf{Gp}_{\mathsf{tf}})$ of extensions (regular epimorphisms) in the category of torsion-free groups. The class $\mathcal{E}$ consists of regular pullback squares, i.e., pullbacks of regular epimorphisms. It is easy to find a split epimorphism of extensions which is not a pullback, and it is also easy to see that (E1)–(E4$^-$) and (E5$^-$) hold using that $\mathsf{Gp}_{\mathsf{tf}}$ is regular Mal'tsev. We give a counterexample for Axiom (F); it is based on the fact that pushouts in $\mathsf{Gp}_{\mathsf{tf}}$ are different from pushouts in Gp and may not be regular pushouts. They are constructed by reflecting the pushout in Gp into the subcategory $\mathsf{Gp}_{\mathsf{tf}}$.

An example of a pushout in $\mathsf{Gp}_{\mathsf{tf}}$ which is not a pushout in Gp is the square

$$\begin{array}{ccc} \mathbb{Z} \times_{\mathbb{Z}_2} \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & 0. \end{array} \qquad (\mathbf{G})$$

($\mathbb{Z}_2$ is torsion while $\mathbb{Z}$ is torsion-free.) The diagram

$$\begin{array}{ccccc} \mathbb{Z} \times_{\mathbb{Z}_2} \mathbb{Z} & \longrightarrow & \mathbb{Z} \times \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ \mathbb{Z} & = & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

now displays a monomorphism composed with an $\mathcal{E}$-extension which cannot be written as an $\mathcal{E}$-extension composed with a monomorphism, as the square ($\mathbf{G}$) is not in $\mathcal{E}$.

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Vakgroep Wiskunde, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium

Institut de Recherche en Mathématique et Physique, Université catholique de Louvain, Chemin du Cyclotron 2 bte L7.01.02, 1348 Louvain-la-Neuve, Belgium

Queens' College, University of Cambridge, United Kingdom

Department of Mathematical Sciences, University of South Africa, Pretoria PO Box 392, UNISA 0003, South Africa

CMUC, University of Coimbra, 3001–454 Coimbra, Portugal

Email: teveraer@vub.ac.be
julia.goedecke@cantab.net
tjanel@gmail.com
tim.vanderlinden@uclouvain.be

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