

NOTE ON STAR-AUTONOMOUS COMONADS

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ABSTRACT. We develop an alternative approach to star-autonomous comonads via linearly distributive categories. It is shown that in the autonomous case the notions of star-autonomous comonad and Hopf comonad coincide.

1. Introduction

Given a linearly distributive category \mathcal{C} , this note determines what structure is required of a comonad G on \mathcal{C} so that \mathcal{C}^G , the category of Eilenberg-Moore coalgebras of G , is again a linearly distributive category. Furthermore, if \mathcal{C} is equipped with negations (and is hence a star-autonomous category), the structure required to lift the negations to \mathcal{C}^G is determined as well. This latter is equivalent to lifting star-autonomy and it is shown that the notion presented is equivalent to a star-autonomous comonad [PS09]. As a consequence of the presentation given here, it may be easily seen that any star-autonomous comonad on an autonomous category is a Hopf monad [BV07].

2. Lifting linear distributivity

Suppose \mathcal{C} is a monoidal category and $G : \mathcal{C} \rightarrow \mathcal{C}$ is a comonad on \mathcal{C} . Recall that \mathcal{C}^G , the category of (Eilenberg-Moore) coalgebras of G , is monoidal if and only if G is a monoidal comonad [M02]. In this section we are interested in the structure required to lift linear distributivity to the category of coalgebras.

A linearly distributive category \mathcal{C} is a category equipped with two monoidal structures (\mathcal{C}, \star, I) and $(\mathcal{C}, \diamond, J)$,¹ and two compatibility natural transformations (called “linear distributions”)

$$\begin{aligned}\partial_l &: A \star (B \diamond C) \rightarrow (A \star B) \diamond C \\ \partial_r &: (B \diamond C) \star A \rightarrow B \diamond (C \star A),\end{aligned}$$

satisfying a large number of coherence diagrams [CS97].

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¹For simplicity we assume that the monoidal structures are strict, although this is not necessary. Furthermore, in their original paper [CS97] the tensor products \star and \diamond are respectively denoted by \otimes and \oplus , and called *tensor* and *par*, emphasizing their connection to linear logic.

Suppose $G = (G, \delta, \epsilon)$ is a comonad on a linearly distributive category \mathcal{C} which is a monoidal comonad on \mathcal{C} with respect to both \star and \diamond , with structure maps (G, ϕ, ϕ_0) and (G, ψ, ψ_0) respectively. If, for G -coalgebras A, B , and C , the comonad G satisfies

$$\begin{array}{ccc} GA \star (GB \diamond GC) & \xrightarrow{1 \star \psi} & GA \star G(B \diamond C) \xrightarrow{\phi} G(A \star (B \diamond C)) \\ \downarrow \partial_l & & \downarrow \partial_l \\ (GA \star GB) \diamond GC & \xrightarrow{\phi \circ 1} & G(A \star B) \diamond GC \xrightarrow{\psi} G((A \star B) \diamond C), \end{array} \quad (1)$$

it may be seen that the morphism ∂_l becomes a G -coalgebra morphism. If G satisfies a similar axiom for ∂_r , i.e.,

$$\begin{array}{ccc} (GB \diamond GC) \star GA & \xrightarrow{\psi \star 1} & G(B \diamond C) \star GA \xrightarrow{\phi} G((B \diamond C) \star A) \\ \downarrow \partial_r & & \downarrow \partial_r \\ GB \diamond (GC \star GA) & \xrightarrow{1 \diamond \phi} & GB \diamond G(C \star A) \xrightarrow{\psi} G(B \diamond (C \star A)), \end{array} \quad (2)$$

then ∂_r also becomes a G -coalgebra morphism. Thus,

2.1. PROPOSITION. *Given a linearly distributive category \mathcal{C} and a comonad $G : \mathcal{C} \rightarrow \mathcal{C}$ satisfying axioms (1) and (2), the category \mathcal{C}^G is a linearly distributive category.*

2.2. EXAMPLE. Let \mathcal{C} be a symmetric linearly distributive category and $(B, \mu, \eta, \delta, \epsilon)$ a bialgebra in \mathcal{C} with respect to \diamond . That is, the structure morphisms are given as

$$\begin{array}{ll} \mu : B \diamond B \rightarrow B & \delta : B \rightarrow B \diamond B \\ \eta : J \rightarrow B & \epsilon : B \rightarrow J. \end{array}$$

Then, $G = B \diamond -$ is a comonad and is monoidal with respect to both \diamond and \star . The latter via $I \cong J \diamond I \xrightarrow{\eta \circ 1} B \diamond I$, and the following,

$$\begin{aligned} (B \diamond U) \star (B \diamond V) & \xrightarrow{\partial_r} B \diamond (U \star (B \diamond V)) \\ & \xrightarrow{1 \diamond (1 \star c)} B \diamond (U \star (V \diamond B)) \\ & \xrightarrow{1 \diamond \partial_l} B \diamond ((U \star V) \diamond B) \\ & \xrightarrow{1 \diamond c} B \diamond (B \diamond (U \star V)) \\ & \xrightarrow{\cong} (B \diamond B) \diamond (U \star V) \\ & \xrightarrow{\mu \star 1} B \diamond (U \star V). \end{aligned}$$

Rather large diagrams, which we leave to the faith of the reader, prove that $B \diamond -$ satisfies (1) and (2), so that $\mathcal{C}^B = \mathbf{Comod}_{\mathcal{C}}(B)$, the category of comodules of B , is a linearly distributive category.

3. Lifting negations

Suppose now that \mathcal{C} is a linearly distributive category equipped with negations S and S' (corresponding to ${}^\perp(-)$ and $(-)^{\perp}$ in [CS97]). That is, functors $S, S' : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ together with the following (dinatural) evaluation and coevaluation morphisms

$$\begin{array}{ccc} SA \star A \xrightarrow{e_A} J & & A \star S'A \xrightarrow{e'_A} J \\ I \xrightarrow{n_A} A \diamond SA & & I \xrightarrow{n'_A} S'A \diamond A, \end{array} \quad (3)$$

satisfying the four evident “triangle identities”. One such is

$$\left(A \cong I \star A \xrightarrow{n \star 1} (A \diamond SA) \star A \xrightarrow{\partial_r} A \diamond (SA \star A) \xrightarrow{1 \circ e} A \diamond J \cong A \right) = 1_A.$$

If \mathcal{C} is equipped with such negations we say simply that \mathcal{C} is a *linearly distributive category with negations*.

We are interested to lift negations to \mathcal{C}^G . This means we must ensure that the “negation” functors $S, S' : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ lift to functors $(\mathcal{C}^G)^{\text{op}} \rightarrow \mathcal{C}^G$, and the evaluation and coevaluation morphisms are in \mathcal{C}^G , i.e., are G -coalgebra morphisms.

The following is essentially known from [S72].

3.1. PROPOSITION. *A (contravariant) functor $S : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ may be lifted to a functor $\tilde{S} : (\mathcal{C}^G)^{\text{op}} \rightarrow \mathcal{C}^G$ such that the diagram*

$$\begin{array}{ccc} (\mathcal{C}^G)^{\text{op}} & \xrightarrow{\tilde{S}} & \mathcal{C}^G \\ U \downarrow & & \downarrow U \\ \mathcal{C}^{\text{op}} & \xrightarrow{S} & \mathcal{C}, \end{array}$$

(in which $U : \mathcal{C}^G \rightarrow \mathcal{C}$ is the underlying functor) commutes, if and only if there is a natural transformation

$$\nu : S \rightarrow GSG$$

satisfying the following two axioms

$$\begin{array}{ccc} \begin{array}{ccc} S & \xrightarrow{\nu} & GSG \\ & \searrow S\epsilon & \downarrow \epsilon_{SG} \\ & & SG \end{array} & & \begin{array}{ccccc} S & \xrightarrow{\nu} & GSG & \xrightarrow{\delta_{SG}} & G^2SG \\ & \searrow \nu & \downarrow G\nu_G & \uparrow G^2S\delta & \\ & & GSG & \xrightarrow{G\nu_G} & G^2SG^2. \end{array} \end{array} \quad (4)$$

This may be viewed as a distributive law of a contravariant functor over a comonad [S72]. In this case, we say that S may be lifted to \mathcal{C}^G , and a functor $\tilde{S} : (\mathcal{C}^G)^{\text{op}} \rightarrow \mathcal{C}^G$ is defined as

$$\tilde{S}(A, \gamma) = \left(SA, SA \xrightarrow{\nu} GSGA \xrightarrow{GS\gamma} GSA \right) \quad \tilde{S}(f) = Sf.$$

(To see the reverse direction, suppose (A, γ) is a coalgebra and \tilde{S} is a functor $\mathcal{C}^G \rightarrow \mathcal{C}^G$, so that $\tilde{S}A = (SA, \tilde{\gamma})$ is again a coalgebra. Define

$$\nu := SA \xrightarrow{\tilde{\gamma}} GSA \xrightarrow{GS\epsilon_A} GSGA,$$

which may be seen to satisfy the axioms in (4).) We will usually let the context differentiate between S and \tilde{S} and simply write S in both cases.

Now, suppose S and S' may be lifted to \mathcal{C}^G , that is, they are equipped respectively with natural transformations

$$\nu : S \rightarrow GSG \quad \text{and} \quad \nu' : S' \rightarrow GS'G,$$

satisfying (4). It remains to lift the evaluation and coevaluation morphisms (3). Consider the following axioms.

$$\begin{array}{ccccc} SA \star GA & \xrightarrow{1 \star \epsilon} & SA \star A & \xrightarrow{e_A} & J \\ \downarrow \nu \star \delta & & & & \downarrow \psi_0 \\ GSGA \star G^2A & \xrightarrow{\phi} & G(SGA \star GA) & \xrightarrow{Ge_{GA}} & GJ \end{array} \quad (5)$$

$$\begin{array}{ccccccc} I & \xrightarrow{\phi_0} & GI & \xrightarrow{Gn} & G(A \diamond SA) & \xrightarrow{G(1 \diamond S\epsilon)} & G(A \diamond SGA) \\ \downarrow n & & & & & & \uparrow G(1 \diamond S\delta) \\ GA \diamond SGA & \xrightarrow{1 \diamond \nu} & GA \diamond GSG^2A & \xrightarrow{\phi} & G(A \diamond SG^2A) & & \end{array} \quad (6)$$

$$\begin{array}{ccccc} GA \star S'A & \xrightarrow{\epsilon \star 1} & A \star S'A & \xrightarrow{e'_A} & J \\ \downarrow \delta \star \nu' & & & & \downarrow \psi_0 \\ G^2A \star GS'GA & \xrightarrow{\phi} & G(GA \star S'GA) & \xrightarrow{Ge'_{GA}} & GJ \end{array} \quad (7)$$

$$\begin{array}{ccccccc} I & \xrightarrow{\phi_0} & GI & \xrightarrow{Gn'} & G(S'A \diamond A) & \xrightarrow{G(S'\epsilon \diamond 1)} & G(S'GA \diamond A) \\ \downarrow n' & & & & & & \uparrow G(S'\delta \diamond 1) \\ S'GA \diamond GA & \xrightarrow{\nu' \diamond 1} & GS'G^2A \diamond GA & \xrightarrow{\phi} & G(S'G^2A \diamond A) & & \end{array} \quad (8)$$

3.2. PROPOSITION. *Suppose \mathcal{C} is a linearly distributive category with negation, G is a monoidal comonad satisfying axioms (1) and (2) (so that \mathcal{C}^G is linearly distributive), and that S and S' may be lifted to \mathcal{C}^G . Then, G satisfies axioms (5), (6), (7), and (8) if and only if \mathcal{C}^G is a linearly distributive category with negation.*

PROOF. Suppose (A, γ) is a G -coalgebra. We start by proving that axiom (5) holds if and only if $e : SA \star A \rightarrow J$ is a G -coalgebra morphism. The following diagram proves the “only if” direction,

$$\begin{array}{ccccccc}
 SA \star A & \xrightarrow{\nu \star \gamma} & GSGA \star GA & \xrightarrow{\phi} & G(SGA \star A) & \xrightarrow{G(S\gamma \star 1)} & G(SA \star A) \\
 \downarrow 1 \star \gamma & & \downarrow 1 \star G\gamma & & \downarrow G(1 \star \gamma) & & \downarrow Ge \\
 SA \star GA & \xrightarrow{\nu \star \delta} & GSGA \star G^2A & \xrightarrow{\phi} & G(SGA \star GA) & \xrightarrow{Ge} & GJ \\
 \downarrow 1 \star \epsilon & & & & & & \downarrow Ge \\
 SA \star A & \xrightarrow{e} & J & \xrightarrow{\psi_0} & GJ & & GJ
 \end{array}
 \quad (5)$$

and this next diagram the “if” direction

$$\begin{array}{ccccccc}
 SA \star GA & \xrightarrow{\nu \star \delta} & GSGA \star G^2A & \xrightarrow{\phi} & G(SGA \star GA) & \xrightarrow{Ge_{GA}} & GJ \\
 \downarrow 1 \star \epsilon & \downarrow S\epsilon \star 1 & \downarrow GSG\epsilon \star 1 & \downarrow 1 & \downarrow Ge_{GA} & & \\
 SA \star A & \xrightarrow{\nu \star \delta} & GSG^2A \star G^2A & \xrightarrow{GS\delta \star 1} & GSGA \star G^2A & \xrightarrow{\phi} & G(SGA \star GA) \\
 \downarrow e & \downarrow e_{GA} & & & & & \downarrow Ge_{GA} \\
 J & \xrightarrow{\psi_0} & J & \xrightarrow{\psi_0} & GJ & & GJ
 \end{array}$$

where the bottom square commutes as e_{GA} is a G -coalgebra morphism.

Next we prove that axiom (6) holds if and only if $n : I \rightarrow A \diamond SA$ is a G -coalgebra morphism. The “only if” direction is given by

$$\begin{array}{ccccccc}
 I & \xrightarrow{\phi_0} & GI & \xrightarrow{Gn} & G(A \diamond SA) & & \\
 \downarrow n & \searrow n & & & \downarrow G(1 \diamond S\epsilon) & & \\
 A \diamond SA & \xrightarrow{1 \diamond \nu} & GA \diamond GSG^2A & \xrightarrow{\phi} & G(A \diamond SG^2A) & \xrightarrow{G(1 \diamond S\delta)} & G(A \diamond SGA) \\
 \downarrow \gamma \diamond 1 & \downarrow 1 \diamond S\gamma & \downarrow 1 \diamond GSG\gamma & \downarrow G(1 \diamond SG\gamma) & \downarrow G(1 \diamond S\gamma) & & \downarrow G(1 \diamond S\gamma) \\
 GA \diamond SA & \xrightarrow{1 \diamond \nu} & GA \diamond GSGA & \xrightarrow{\phi} & G(A \diamond SGA) & \xrightarrow{G(1 \diamond S\gamma)} & G(A \diamond SA)
 \end{array}
 \quad (6)$$

and the “if” direction by

$$\begin{array}{c}
 I \xrightarrow{\phi_0} GI \\
 \downarrow n_{GA} \qquad \qquad \qquad \downarrow Gn_{GA} \qquad \searrow Gn \\
 GA \diamond SGA \xrightarrow{\delta \diamond \nu} G^2 A \diamond GSG^2 A \xrightarrow{\psi} G(GA \diamond SG^2 A) \xrightarrow{G(1 \diamond S\delta)} G(GA \diamond SGA) \quad G(A \diamond SA) \\
 \searrow 1 \diamond \nu \qquad \downarrow G\epsilon \diamond 1 \qquad \downarrow G(\epsilon \diamond 1) \qquad \downarrow G(\epsilon \diamond 1) \qquad \swarrow G(1 \diamond S\epsilon) \\
 GA \diamond GSG^2 A \xrightarrow{\psi} G(A \diamond SG^2 A) \xrightarrow{G(1 \diamond S\delta)} G(A \diamond SGA),
 \end{array}$$

where the top square commutes as n_{GA} is a G -coalgebra morphism.

The remaining two axioms are proved similarly. ■

4. Star-autonomous comonads

Suppose $\mathcal{C} = (\mathcal{C}, \otimes, I, S, S')$ is a star-autonomous category. A star-autonomous comonad $G : \mathcal{C} \rightarrow \mathcal{C}$ is a comonad satisfying axioms (described below) so that \mathcal{C}^G becomes a star-autonomous category [PS09]. In this section we show that comonads as in Proposition 3.2 and star-autonomous comonads coincide.

We recall the definition of star-autonomous comonad [PS09], but, as it suits our needs better here, we present a more symmetric version. First recall that a star-autonomous category $\mathcal{C} = (\mathcal{C}, \otimes, I, S, S')$ may be defined as a monoidal category $(\mathcal{C}, \otimes, I)$ equipped with an adjoint equivalence

$$S \dashv S' : \mathcal{C}^{\text{op}} \xrightarrow{\cong} \mathcal{C}$$

such that

$$\mathcal{C}(A \otimes B, SC) \cong \mathcal{C}(A, S(B \otimes C)), \quad (9)$$

natural in $A, B, C \in \mathcal{C}$. The functor S is called the *left star operation* and S' the *right star operation*.

By the Yoneda lemma, the isomorphism in (9) determines, and is determined by, the two following “evaluation” morphisms:

$$e = e_{A,B} : S(A \otimes B) \otimes A \rightarrow SB \quad \text{and} \quad e' = e'_{B,A} : B \otimes S'(A \otimes B) \rightarrow S'A.$$

4.1. DEFINITION. A star-autonomous comonad on a star-autonomous category \mathcal{C} is a monoidal comonad $G : \mathcal{C} \rightarrow \mathcal{C}$ equipped with

$$\nu : S \rightarrow GSG \quad \text{and} \quad \nu' : S' \rightarrow GS'G,$$

satisfying (4) (i.e., S and S' may be lifted to \mathcal{C}^G), and this data must be such that the following four diagrams commute.

$$\begin{array}{ccc}
SS'G & \xrightarrow{\cong} & G \\
\downarrow \nu & & \downarrow \cong \\
GSGS'G & \xrightarrow{GS\nu'} & GSS'
\end{array}
\qquad
\begin{array}{ccc}
S'SG & \xrightarrow{\cong} & G \\
\downarrow \nu' & & \downarrow \cong \\
GS'GSG & \xrightarrow{GS\nu} & GS'S
\end{array}$$

$$\begin{array}{ccccc}
S(A \otimes B) \otimes GA & \xrightarrow{1 \otimes \epsilon} & S(A \otimes B) \otimes A & \xrightarrow{e_{A,B}} & SB \\
\downarrow \nu \otimes \delta & & & & \downarrow \nu \\
GSG(A \otimes B) \otimes G^2A & & & & GSSB \\
\downarrow \phi & & & & \uparrow Ge_{GA,GB} \\
G(SG(A \otimes B) \otimes GA) & \xrightarrow{G(S\phi \otimes 1)} & G(S(GA \otimes GB) \otimes GA) & &
\end{array}$$

$$\begin{array}{ccccc}
GB \otimes S'(A \otimes B) & \xrightarrow{\epsilon \otimes 1} & B \otimes S'(A \otimes B) & \xrightarrow{e'_{B,A}} & S'A \\
\downarrow \delta \otimes \nu' & & & & \downarrow \nu' \\
G^2B \otimes GS'G(A \otimes B) & & & & GS'GA \\
\downarrow \phi & & & & \uparrow Ge'_{GB,GA} \\
G(GB \otimes S'G(A \otimes B)) & \xrightarrow{G(1 \otimes S'\phi)} & G(GB \otimes S'(GA \otimes GB)) & &
\end{array}$$

The first two diagrams above ensure that the unit and the counit of the adjoint equivalence $S \dashv S'$ lifts to \mathcal{C}^G , while the latter two diagrams above respectively ensure that e and e' are G -coalgebra morphisms, so that the isomorphism (9) also lifts to \mathcal{C}^G .

We wish to show that star-autonomous comonads and comonads as in Proposition 3.2 coincide. It should not be surprising considering the following theorem.

4.2. THEOREM. [CS97, Theorem 4.5] *The notions of linearly distributive categories with negation and star-autonomous categories coincide.*

Given a star-autonomous category $(\mathcal{C}, \otimes, I_\otimes, S, S')$, identifying $\star := \otimes$ (and the monoidal unit $I := I_\star = I_\otimes$) and defining

$$A \diamond B := S'(SB \star SA) \cong S(S'B \star S'A) \qquad J := SI \cong S'I \qquad (10)$$

results in a linearly distributive category with negations $(\mathcal{C}, \star, I, \diamond, J, S, S')$ [CS97]. In [CS97], they consider the symmetric case, but the correspondence between linearly distributive categories with negation and star-autonomous categories holds in the noncommutative case as well.

Given Theorem 4.2, Proposition 3.2 says that if \mathcal{C} is star-autonomous, and G is such a comonad, then \mathcal{C}^G is star-autonomous. We now compare the two definitions.

Suppose now that G is a comonad on a linearly distributive category \mathcal{C} , as in Proposition 3.2. We wish to show that it is a star-autonomous comonad. Rather than proving the axioms, it is simpler to show directly that the morphisms under consideration are G -coalgebra morphisms. To this end, the unit and the counit of the adjoint equivalence $S \dashv S'$ are defined respectively by the composites

$$A \cong I \star A \xrightarrow{n'_{SA} \star 1} (S' SA \diamond SA) \star A \xrightarrow{\partial_r} S' SA \diamond (SA \star A) \xrightarrow{1 \diamond n} S' SA \diamond J \cong S' SA$$

and

$$S' SA \cong I \star S' SA \xrightarrow{n_{SA} \star 1} (A \diamond SA) \star S' SA \xrightarrow{\partial_r} A \diamond (SA \star S' SA) \xrightarrow{1 \diamond e'_{SA}} A \diamond J \cong A,$$

and the evaluation morphisms $e_{A,B}$ and $e'_{B,A}$ respectively by the composites

$$\begin{array}{ccc} S(A \star B) \star A & \xrightarrow{\quad e_{A,B} \quad} & SB \\ \cong \downarrow & & \cong \uparrow \\ (S(A \star B) \star A) \star I & & J \diamond SB \\ \downarrow 1 \star n & & \uparrow e_{A \star B \diamond 1} \\ (S(A \star B) \star A) \star (B \diamond SB) & \xrightarrow{\quad \partial_l \quad} & (S(A \star B) \star A \star B) \diamond SB, \end{array}$$

and

$$\begin{array}{ccc} B \star S'(A \star B) & \xrightarrow{\quad e'_{B,A} \quad} & SB \\ \cong \downarrow & & \cong \uparrow \\ I \star (B \star S'(A \star B)) & & S' A \diamond J \\ \downarrow n' \star 1 & & \uparrow 1 \diamond e'_{A \star B} \\ (S' A \diamond A) \star (B \star S'(A \star B)) & \xrightarrow{\quad \partial_r \quad} & S' A \diamond (A \star B \star S'(A \star B)). \end{array}$$

In the situation of Proposition 3.2, we see that all four of these morphisms are given as composites of G -coalgebra morphisms, and thus, are G -coalgebra morphisms themselves. Therefore, G is a star-autonomous comonad.

In the other direction suppose G is a star-autonomous comonad on a star-autonomous category \mathcal{C} . It is similar to show that it is a comonad satisfying the requirements of Proposition 3.2. Using the identifications in (10), the two linear distributions are defined as the following composites.

$$\begin{array}{ccc}
A \star (B \diamond C) & \xrightarrow{\partial_l} & (A \star B) \diamond C \\
\cong \downarrow & & \uparrow \cong \\
A \otimes S'(SC \otimes SB) & & S'(SC \otimes S(A \otimes B)) \\
1 \otimes S'(1 \otimes e) \searrow & & \nearrow e' \\
A \otimes S'(SC \otimes S(A \otimes B) \otimes A) & &
\end{array}
\qquad
\begin{array}{ccc}
(B \diamond C) \star A & \xrightarrow{\partial_r} & B \diamond (C \star A) \\
\cong \downarrow & & \uparrow \cong \\
S(S'C \otimes S'B) \otimes A & & S(S'(C \otimes A) \otimes S'B) \\
S(e' \otimes 1) \otimes 1 \searrow & & \nearrow e \\
S(A \otimes S'(C \otimes A) \otimes S'B) \otimes A & &
\end{array}$$

The evaluation maps e_A and e'_A are defined as $e_{A,I}$ and $e'_{A,I}$, and the coevaluation maps n_A and n'_A as

$$\begin{aligned}
n_A &= \left(I \cong SS'I \xrightarrow{S(e'_{A,I})} S(A \otimes S'A) = A \diamond SA \right) \\
n'_A &= \left(I \cong S'SI \xrightarrow{S'(e_{A,I})} S'(SA \otimes A) = S'A \diamond A \right)
\end{aligned}$$

Again, each morphism is a G -coalgebra morphism, or composite thereof, and therefore is itself a G -coalgebra morphism.

Thus, both notions coincide, and we will simply call either a *star-autonomous comonad*, and let context differentiate the axiomatization.

4.3. EXAMPLE. Any Hopf algebra H in a star-autonomous category \mathcal{C} gives rise to a star-autonomous comonad $H \otimes - : \mathcal{C} \rightarrow \mathcal{C}$. See [PS09, p. 3515] for details.

4.4. EXAMPLE. If \mathcal{C} is a symmetric closed monoidal category with finite products, then we may apply an instance of the Chu construction [B79] to produce a star-autonomous category $\text{Chu}(\mathcal{C}, 1)$. The category \mathcal{C} fully faithfully embeds into $\text{Chu}(\mathcal{C}, 1)$,

$$\mathcal{C} \hookrightarrow \text{Chu}(\mathcal{C}, 1)$$

and this functor is strong symmetric monoidal. Thus, any Hopf algebra in \mathcal{C} becomes a Hopf algebra in $\text{Chu}(\mathcal{C}, 1)$, and thus, an example of a star-autonomous comonad.

5. The compact case $\star = \diamond$

If \mathcal{C} is a linearly distributive category with negation for which $\star = \diamond$ (and thus, $I = J$), then \mathcal{C} is an autonomous (= rigid) category. The functor S provides left duals, while S' provides right duals. It is not hard to see that, in this case, any star-autonomous monad G (after turning arrows around) is a Hopf monad [BV07]. Set $\star = \diamond$ and $I = J$ and

dualize axioms (5), (6), (7), and (8). They correspond in [BV07] to axioms (23), (22), (21), and (20) respectively. (In their notation ${}^{\vee}(-) = S$ and $(-)^{\vee} = S'$.) Therefore, we have:

5.1. PROPOSITION. *Star-autonomous monads on autonomous categories are Hopf monads.*

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