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ON PROPERTY-LIKE STRUCTURES

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ABSTRACT. A category may bear many monoidal structures, but (to within a unique isomorphism) only one structure of "category with finite products". To capture such distinctions, we consider on a 2-category those 2-monads for which algebra structure is essentially unique if it exists, giving a precise mathematical definition of "essentially unique" and investigating its consequences. We call such 2-monads *property-like.* We further consider the more restricted class of *fully property-like* 2-monads, consisting of those property-like 2-monads for which all 2-cells between (even lax) algebra morphisms are algebra 2-cells. The consideration of lax morphisms leads us to a new characterization of those monads, studied by Kock and Zöberlein, for which "structure is adjoint to unit", and which we now call *lax-idempotent* 2-monads: both these and their *colax-idempotent* duals are fully property-like. We end by showing that (at least for finitary 2-monads) the classes of property-likes, fully property-likes, and lax-idempotents are each coreflective among all 2-monads.

1. Introduction

A monoidal category is an example of a "category with extra structure of an algebraic kind", in that it is an algebra for a certain 2-monad T on the 2-category **Cat**, and is thus given by its (underlying) category A together with an *action* $a : TA \to A$ of T on A in the usual strict sense; this action encodes the extra — that is, the monoidal — structure given by the tensor product $\otimes$, the unit object I, and the various structure-isomorphisms, subject to Mac Lane's coherence conditions. Of course, a given category A may admit many such monoidal structures.

Another example of a category with "algebraic extra structure" is given by a category with finite coproducts. Here the action $a : TA \to A$ (for a different 2-monad T) encodes the coproduct structure, including the coprojections and so on. This time, however, in contrast to the first example, the structure is uniquely determined (when it exists) to within appropriate isomorphisms — indeed, to within *unique* such isomorphisms; so that to give an A with such a structure is just to give an A with a certain *property* — in this case, the property of admitting finite coproducts.

In fact the notion of "algebraic" extra structure on a category is somewhat wider than that of "algebra for a 2-monad on **Cat**"; monoidal closed categories, for instance, are the algebras for a 2-monad on the 2-category $\mathbf{Cat}_g$ of categories, functors, and natural *isomorphisms*, but not for any 2-monad on **Cat** — see [5, Section 6]. Again, the underlying

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object may be not a single category but a family of categories, or even a diagram in **Cat** — so that here T is to be a 2-monad on some power $\mathbf{Cat}^X$ or some functor 2-category $[\mathcal{A}, \mathbf{Cat}]$, or perhaps $[\mathcal{A}, \mathbf{Cat}_g]$. To capture such cases, we place ourselves in the general context of a 2-monad $T = (T, m, i)$ on a 2-category $\mathcal{K}$ (which at times we suppose to have various completeness or cocompleteness properties). We mean (T, m, i) here to be a 2-monad in the *strict* sense, given by a 2-functor $T : \mathcal{K} \to \mathcal{K}$ together with 2-natural transformations $m : T^2 \to T$ and $i : 1 \to T$ satisfying $m.mT = m.Tm$, $m.iT = 1_T$, and $m.Ti = 1_T$. This generality suffices because, on the one hand, it follows from the analysis of [10], carried out more formally in [20], that the usual "algebraic" structures are indeed algebras for such a strict T; and, on the other, various coherence results allow us to reduce the study of a "pseudo" monad to that of a strict one. As we said, our actions $a : TA \to A$ are also strict, in the sense that they satisfy $a.mA = a.Ta$ and $a.iA = 1_A$; and a T-algebra (A, a) is an $A \in \mathcal{K}$ along with such an action a.

As for more general structures that are not monadic at all, but may be models, say, of a two-dimensional finite-limit-theory — such as the extensive categories of [7] — these must wait for later investigations: needing to begin somewhere, we have chosen to start with the simple monadic (or "purely algebraic") case.

In an example so simple as that of finite coproducts, we know precisely in what sense the structure is "unique to within a unique isomorphism"; but it is not so obvious what such uniqueness should mean in the case of a general 2-monad T on a 2-category $\mathcal{K}$, even in the case where $\mathcal{K}$ is just **Cat**. Our first goal is to provide a useful definition in this general setting (comparing it with possible alternative or stronger forms) and to deduce mathematical consequences of a 2-monad's having this "uniqueness of structure" property, or variants thereof.

In doing so, we are led to consider not only the algebras, but also their morphisms. Recall, for the cases both of monoidal categories and of categories with finite coproducts, that the morphisms of chief practical interest are not those which preserve the structure "on the nose", but rather those which preserve it to within (suitably coherent) isomorphisms; it is these that we shall call *T-morphisms*, the detailed definition being given below. We shall use the name *strict T-morphism* for those preserving the structure on the nose; they retain a certain theoretical importance, as being the morphisms of the "Eilenberg-Moore object". Various authors [11,2,26] have also pointed out the importance of *lax* morphisms, and these too will play a prominent role in our analysis; we define a *lax T-morphism* from a T-algebra (A, a) to a T-algebra (B, b) to be a pair $(f, \bar{f})$, where $f : A \to B$ is a morphism in $\mathcal{K}$, and $\bar{f}$ is a 2-cell, not necessarily invertible, as in

$$\begin{array}{ccc} TA & \xrightarrow{\;a\;} & A \\ {\scriptstyle Tf}\big\downarrow & \overset{\bar{f}}{\Rightarrow} & \big\downarrow{\scriptstyle f} \\ TB & \xrightarrow[\;b\;]{} & B \end{array},$$

which satisfies the following "coherence" conditions:

$$\begin{array}{ccccc} T^2A & \xrightarrow{m_A} & TA & \xrightarrow{a} & A \\ {\scriptstyle T^2f}\downarrow & & {\scriptstyle Tf}\downarrow & \overset{\bar f}{\Rightarrow} & \downarrow{\scriptstyle f} \\ T^2B & \xrightarrow[m_B]{} & TB & \xrightarrow[b]{} & B \end{array} \quad = \quad \begin{array}{ccccc} T^2A & \xrightarrow{Ta} & TA & \xrightarrow{a} & A \\ {\scriptstyle T^2f}\downarrow & \overset{T\bar f}{\Rightarrow} & {\scriptstyle Tf}\downarrow & \overset{\bar f}{\Rightarrow} & \downarrow{\scriptstyle f} \\ T^2B & \xrightarrow[Tb]{} & TB & \xrightarrow[b]{} & B \end{array}$$

and

$$\begin{array}{ccccc} A & \xrightarrow{i_A} & TA & \xrightarrow{a} & A \\ {\scriptstyle f}\downarrow & & {\scriptstyle Tf}\downarrow & \overset{\bar f}{\Rightarrow} & \downarrow{\scriptstyle f} \\ B & \xrightarrow[i_B]{} & TB & \xrightarrow[b]{} & B \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[1_B]{} & B. \end{array}$$

(Note that regions in which no 2-cell is written are always commutative, and are deemed to contain the identity 2-cell.) Now we can define the T-morphisms precisely, as being those lax ones $(f, \bar f)$ for which $\bar f$ is invertible; while the strict T-morphisms are those $(f, \bar f)$ for which $\bar f$ is an identity — or equivalently, just those arrows $f : A \to B$ for which $b.Tf = f.a$. Of course there is also the dual notion of *colax T-morphism*, in which the sense of the 2-cell $\bar f$ is reversed. When it is clear which T is meant, we may write *algebra morphism* for T-morphism.

Using this notion of T-morphism, we can now express more precisely what it might mean to say that an action of T on A is "unique to within a unique isomorphism": we shall mean that, given two actions $a, a' : TA \to A$, there is a unique isomorphism $\theta : a' \to a$ for which $(1_A, \theta) : (A, a) \to (A, a')$ is a T-morphism (and hence, by a simple argument, an *isomorphism* of T-algebras). For such a T, we may say for short that *T-algebra structure is essentially unique.*

The matter of T-morphisms reveals another difference between our first two examples. In the case of a functor $f : A \to B$ between categories with finite coproducts, there are the canonical comparisons $fx + fy \to f(x + y)$ and $0_B \to f(0_A)$; and it turns out that there is some algebra morphism $(f, \bar f) : A \to B$ if and only if these canonical morphisms are invertible — that is, if and only if f "preserves coproducts" in the usual sense; moreover in this case the $\bar f$ is unique. On the other hand, in the case of monoidal categories A and B, to give $\bar f$ amounts to giving isomorphisms $\tilde f_2 : fx \otimes fy \cong f(x \otimes y)$ and $\tilde f_0 : I_B \cong f(I_A)$ satisfying naturality and coherence conditions; so that here an algebra morphism $(f, \bar f)$ involves an underlying functor f and *extra structure* in the form of $\bar f$, this extra structure being by no means uniquely determined. We may say that *T-morphism structure is unique* if, given T-algebras (A, a) and (B, b) and given $f : A \to B$ in $\mathcal{K}$, there is at most one $\bar f$ for which $(f, \bar f)$ is a T-morphism.

We could consider a stronger version of the essential uniqueness of T-algebra structure, imposing it not only for actions $a : TA \to A$ of T on an object A of $\mathcal{K}$, but also for actions $x : TX \to X$ of T on a 2-functor $X : \mathcal{C} \to \mathcal{K}$ of codomain $\mathcal{K}$. Whether this stronger version is in fact *strictly* stronger is unknown to us, but it is satisfied for any $\mathcal{C}$ if it is satisfied for $\mathcal{C} = \mathbb{2}$, the "arrow" category, seen as a 2-category with only identity 2-cells. Moreover, we shall see that this stronger version implies uniqueness of T-morphism structure, and is equivalent to the weaker version augmented by this latter. Accordingly we call T *property-like* when T-algebra structure is essentially unique in this stronger sense; which is equally to say that T has *both* essential uniqueness of algebra structure (in the original sense) *and* uniqueness of morphism structure.

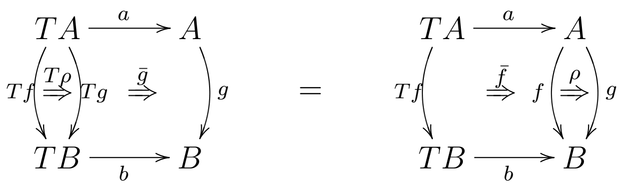
Thus the 2-monad for categories with finite coproducts is property-like; and so dually is that for categories with finite products. In fact the first of these has a still stronger property, which the second lacks (although possessing its dual): namely, given algebras (A, a) and (B, b) and any morphism $f : A \to B$ in $\mathcal{K}$, there is a unique $\bar{f}$ for which $(f, \bar{f})$ is a lax morphism of algebras $(A, a) \to (B, b)$. (In the case of finite coproducts, $\bar{f}$ is induced by the canonical comparisons $fx + fy \to f(x + y)$ and $0_B \to f(0_A)$; and it is *this* $(f, \bar{f})$ which is a T-morphism when f preserves coproducts in the usual sense.) We shall call a 2-monad $T = (T, m, i)$ with this property *lax-idempotent*; and we shall see in Section 6 below that such 2-monads are precisely those, associated with the names of Kock and Zöberlein (see [22] and [28]), for which "structure is adjoint to unit", in the sense that there is, in the functor 2-category $[\mathcal{K}, \mathcal{K}]$, an adjunction $m \dashv iT$ whose counit is the identity 2-cell $m.iT = 1_T$. The prime examples of structures given by lax-idempotent 2-monads are categories with colimits of some class (such as the categories with finite coproducts we have been considering). There is also the dual case where m is *right* adjoint to iT, the prime examples of such structures being categories with limits of some class; these 2-monads are called *colax-idempotent*, and they can equally be characterized by the existence and uniqueness of colax T-morphism structure. We shall see that both lax-idempotent 2-monads and colax-idempotent 2-monads are always property-like, while so too are such 2-monads as that for distributive categories, which involves both colimit structure and limit structure, but is itself neither lax-idempotent nor colax-idempotent. Indeed this 2-monad may be formed as a quotient

$$T + S \xrightarrow{q} D$$

in the 2-category 2-Mnd(**Cat**) of 2-monads on **Cat**, described in Section 2 below; here T is the 2-monad for categories with finite coproducts, S is the 2-monad for categories with finite products, $T + S$ is the coproduct in 2-Mnd(**Cat**), and q is a certain coinverter and hence is epimorphic and *co-fully-faithful*, this last meaning that it is representably fully faithful when seen as an arrow of 2-Mnd$(\mathbf{Cat})^{op}$. (In future, when $q : X \to D$ is epimorphic in some 2-category, we shall call D a *quotient* of X; and a *co-fully-faithful quotient* if moreover q is co-fully-faithful.) Since we shall see in Remark 4.3 below that the property-likes are closed in 2-Mnd(**Cat**) under colimits and under co-fully-faithful quotients such as q, it follows that D is property-like.

Of course lax T-morphisms compose to form a category (with subcategories provided by the T-morphisms and the strict T-morphisms). In fact (see [5]) the category of T-algebras and lax T-morphisms becomes a 2-category $T\text{-Alg}_l$ when we introduce as 2-cells the *T-transformations*, where a T-transformation from $(f, \bar{f}) : (A, a) \to (B, b)$ to

$(g, \bar{g}) : (A, a) \to (B, b)$ is a 2-cell $\rho : f \to g$ in $\mathcal{K}$ satisfying the single "coherence" condition



expressing compatibility of ρ with $\bar{f}$ and $\bar{g}$. We further write T-Alg for the locally-full sub-2-category of T-Alg$_l$ determined by the T-morphisms, and T-Alg$_s$ for the locally-full sub-2-category determined by the strict T-morphisms. Similarly there is a notion of T-transformation between *colax* T-morphisms, and this gives a 2-category T-Alg$_c$ of T-algebras, colax T-morphisms, and T-transformations. There are evident forgetful 2-functors $U_l : T\text{-Alg}_l \to \mathcal{K}$, $U : T\text{-Alg} \to \mathcal{K}$, and $U_c : T\text{-Alg}_c \to \mathcal{K}$ (apart from the Eilenberg-Moore 2-functor $U_s : T\text{-Alg}_s \to \mathcal{K}$).

This notion of T-transformation underlies yet another difference between the structure of a monoidal category and that of a category with finite coproducts. In the case where T is the 2-monad on **Cat** whose algebras are monoidal categories, the above definition of T-transformation gives precisely the usual notion [11] of *monoidal natural transformation*; on the other hand one does not speak of a "finite-coproduct-compatible-natural-transformation" because, when the structure involved is finite coproducts, the coherence condition for a T-transformation becomes vacuous: given a parallel pair of lax T-morphisms $(f, \bar{f})$ and $(g, \bar{g})$, *every* 2-cell $\rho : f \to g$ is a T-transformation. It is shown in Proposition 5.2 below that this vacuousness does not hold for a general property-like T; we shall call a property-like T for which it does hold *fully property-like*. Among the fully property-like 2-monads are the lax-idempotent ones, and hence by duality the colax-idempotent ones — as well as co-fully-faithful quotients of colimits of these, since the fully property-likes are again closed under (at least conical) colimits and co-fully-faithful quotients in 2-Mnd(**Cat**).

An ordinary category $\mathcal{K}$ may be seen as a locally-discrete 2-category — that is, one whose only 2-cells are identities; and then any monad T on $\mathcal{K}$ may be seen as a 2-monad. For such a T, of course, every lax T-morphism is a strict one. To say that this 2-monad T is lax-idempotent is of course to say that, for algebras (A, a) and (B, b), every $f : A \to B$ in $\mathcal{K}$ is a (strict) T-morphism; this is in turn to say that T is an *idempotent* monad, corresponding to the reflection of $\mathcal{K}$ onto some full subcategory. It is interesting to note that the ordinary monad T, seen as a 2-monad, may well be property-like without being idempotent. For one easily verifies that the ordinary monad T is property-like as a 2-monad precisely when the forgetful functor $U^T : T\text{-Alg} \to \mathcal{K}$ is injective on objects; yet the monadic forgetful functor $U : \mathbf{Mon} \to \mathbf{Sgrp}$ from the category of monoids to that of semigroups is injective on objects, but not fully faithful; so that the corresponding monad on **Sgrp** is property-like without being idempotent.

The plan of the paper is as follows. In Section 2 we recall the Kan extension techniques which, when $\mathcal{K}$ admits the appropriate limits, allow us to replace T-actions $a : TA \to A$ by monad morphisms $\alpha : T \to \langle A, A \rangle$ and to replace T-morphisms $(f, \bar{f}) : (A, a) \to (B, b)$

by monad morphisms $\gamma : T \to \{f, f\}$. In Section 3 we introduce a general framework for discussing those properties of 2-monads concerned with existence or uniqueness of T-morphism structure; among such properties are several of those introduced above. Using these preliminaries, we examine property-like 2-monads in Section 4, fully property-like ones in Section 5, and lax-idempotent ones in Section 6. After some comments in Section 7 on the particular case of a mere category $\mathcal{K}$, we show in Section 8 that our various special classes of 2-monads are closed in 2-Mnd($\mathcal{K}$) under colimits (and certain quotients); whence we are led to investigate how far these classes are coreflective in 2-Mnd($\mathcal{K}$).

2. Morphisms of monads and actions of monads

It is convenient to make some very general observations on *actions*; many of them are well known, and may be considered as folklore. First, given a monoidal category $\mathcal{M} = (\mathcal{M}, \otimes, I)$, by a *monoid* in $\mathcal{M}$ we mean an object T along with morphisms $m : T \otimes T \to T$ and $i : I \to T$ satisfying the associativity and two-sided unit laws; these form a category Mon($\mathcal{M}$) when we take a *monoid morphism* to be a morphism $\alpha : T \to S$ in $\mathcal{M}$ for which $n(\alpha \otimes \alpha) = \alpha m$ and $j = \alpha i$.

A general context in which one speaks of *actions* is the following. We begin with a category $\mathcal{A}$, a monoidal category $\mathcal{M} = (\mathcal{M}, \otimes, I)$, and an *action of* $\mathcal{M}$ *on* $\mathcal{A}$: this last is a functor $\diamond : \mathcal{M} \times \mathcal{A} \to \mathcal{A}$, along with natural isomorphisms $(T \otimes S) \diamond A \cong T \diamond (S \diamond A)$ and $I \diamond A \cong A$ satisfying pentagonal and triangular coherence conditions resembling those of Mac Lane for monoidal categories. Now, for a monoid $T = (T, m, i)$ in $\mathcal{M}$ and an object A of $\mathcal{A}$, we have the notion of an *action of* T *on* A: namely, a morphism $a : T \diamond A \to A$ satisfying the usual associativity and unit laws. In our applications, $\mathcal{M}$ is always a *strict* monoidal category and $\diamond$ a *strict* action of $\mathcal{M}$ on $\mathcal{A}$; that is, one for which the isomorphisms above are equalities of functors, given on objects by $(T \otimes S) \diamond A = T \diamond (S \diamond A)$ and $I \diamond A = A$; and in these circumstances we are led to replace $T \otimes S$ by the simpler TS, and $T \diamond A$ by TA, with 1 for the unit I. Of course an $A \in \mathcal{A}$ with such an action $a : TA \to A$ is called a T-algebra, and we have the usual notion of (strict) morphism of T-algebras.

It may be that, for each $A \in \mathcal{A}$, the functor $- \diamond A : \mathcal{M} \to \mathcal{A}$ sending T to TA has a right adjoint $\langle A, - \rangle : \mathcal{A} \to \mathcal{M}$; then $\langle A, B \rangle$ is of course the value on objects of a functor $\langle -, - \rangle : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{M}$, and we have a natural isomorphism

$$\pi_{T,A,B} : \mathcal{A}(TA, B) \cong \mathcal{M}(T, \langle A, B \rangle).$$

Thus π gives a bijection between morphisms $a : TA \to B$ and morphisms $\alpha : T \to \langle A, B \rangle$ in $\mathcal{M}$; and a here is given explicitly in terms of α as the composite

$$TA \xrightarrow{\alpha A} \langle A, B \rangle A \xrightarrow{\epsilon_{A,B}} B \ ,$$

where $\epsilon_{A,B}$ denotes the counit of the adjunction π.

If we now take $B = A$, the morphism

$$\langle A, A \rangle \langle A, A \rangle A \xrightarrow{\langle A, A \rangle \epsilon_{A,A}} \langle A, A \rangle A \xrightarrow{\epsilon_{A,A}} A$$

gives, on applying π, a "multiplication" $n : \langle A, A\rangle\langle A, A\rangle \to \langle A, A\rangle$, while the identity morphism $1A \to A$ gives, on applying π, a "unit" $j : 1 \to \langle A, A\rangle$. One easily verifies, first, that $(\langle A, A\rangle, n, j)$ is a monoid in $\mathcal{M}$; and next, that if T too is a monoid (T, m, i) in $\mathcal{M}$, then the morphism $a : TA \to A$ is an *action* of T on A precisely when $\alpha : T \to \langle A, A\rangle$ is a monoid morphism.

In our applications below, $\mathcal{A}$ is in fact a 2-category, $\mathcal{M}$ is a monoidal 2-category, and the action $\mathcal{M} \times \mathcal{A} \to \mathcal{A}$ is a 2-functor. Here $\mathrm{Mon}(\mathcal{M})$ is a 2-category, since besides the notion of a monoid morphism $\alpha : (T, m, i) \to (S, n, j)$ we have that of a *monoid transformation* $\chi : \alpha \to \beta : (T, m, i) \to (S, n, j)$: namely a 2-cell $\chi : \alpha \to \beta$ in $\mathcal{M}$ for which (still denoting $\otimes$ by concatenation) we have

$$TT \underset{\beta\beta}{\overset{\alpha\alpha}{\underset{\Downarrow\chi\chi}{\rightrightarrows}}} SS \xrightarrow{\;n\;} S \qquad = \qquad TT \xrightarrow{\;m\;} T \underset{\beta}{\overset{\alpha}{\underset{\Downarrow\chi}{\rightrightarrows}}} S$$

and

$$1 \underset{j}{\overset{j}{\underset{\Downarrow 1}{\rightrightarrows}}} S \qquad = \qquad 1 \xrightarrow{\;i\;} T \underset{\beta}{\overset{\alpha}{\underset{\Downarrow\chi}{\rightrightarrows}}} S\ .$$

Note that our monoid morphisms α are still required to satisfy $n.\alpha\alpha = \alpha.m$ and $j = \alpha i$ on the nose; this is appropriate because we are still interested in *strict* actions $a : TA \to A$ and the corresponding T-algebras (A, a); between such algebras, however we are now in a position to consider *lax* T-morphisms and so on.

It is further the case in our applications below that $\langle A, -\rangle$ is right adjoint to $- \diamond A$ *as a 2-functor*, so that $\langle -, -\rangle$ is now not just a functor but a 2-functor, while $\pi_{T,A,B} : \mathcal{A}(TA, B) \cong \mathcal{M}(T, \langle A, B\rangle)$ is a 2-natural isomorphism of *categories*, taking a 2-cell $\theta : a_1 \to a_2 : TA \to B$ in $\mathcal{A}$ to a 2-cell $\chi : \alpha_1 \to \alpha_2 : T \to \langle A, B\rangle$ in $\mathcal{M}$. Now, in the case $B = A$, if $a_1, a_2 : TA \to A$ are actions of T on A, so that the corresponding $\alpha_1, \alpha_2 : T \to \langle A, A\rangle$ are monoid morphisms, one easily verifies that the 2-cell $\chi : \alpha_1 \to \alpha_2$ is a monoid transformation if and only if the corresponding 2-cell $\theta : a_1 \to a_2$ makes $(1, \theta) : (A, a_2) \to (A, a_1)$ into a lax T-morphism.

For our first application we take for $\mathcal{A}$ the 2-category $\mathcal{K}$ of our earlier considerations, and take for $\mathcal{M}$ the 2-category $[\mathcal{K}, \mathcal{K}]$ of endo-2-functors, 2-natural transformations, and modifications, which becomes a *monoidal* 2-category when we take the tensor-product 2-functor $[\mathcal{K}, \mathcal{K}] \times [\mathcal{K}, \mathcal{K}] \to [\mathcal{K}, \mathcal{K}]$ to be *composition*, given on objects by $(T, S) \mapsto TS$, the identity object for which is the identity functor $1_{\mathcal{K}}$. The monoids in $[\mathcal{K}, \mathcal{K}]$ are of course the 2-monads on $\mathcal{K}$; the monoid morphisms $\alpha : T \to S$, which are the 2-natural transformations satisfying $n.\alpha\alpha = \alpha.m$ and $j = \alpha i$, are here called the *monad morphisms*; and the monoid transformations, which are the modifications χ satisfying $n.\chi\chi = \chi.m$ and $j = \chi i$, are called the *monad modifications* — note that $\chi\chi : \alpha\alpha \to \beta\beta : TT \to SS$ here denotes the common value in

$$TT \underset{T\beta}{\overset{T\alpha}{\underset{\Downarrow T\chi}{\rightrightarrows}}} TS \underset{\beta S}{\overset{\alpha S}{\underset{\Downarrow\chi S}{\rightrightarrows}}} SS \qquad = \qquad TT \underset{\beta T}{\overset{\alpha T}{\underset{\Downarrow\chi T}{\rightrightarrows}}} ST \underset{S\beta}{\overset{S\alpha}{\underset{\Downarrow S\chi}{\rightrightarrows}}} SS\ ,$$

where for instance $T\chi$ has components $(T\chi)_A = T\chi_A$ and χS has components $(\chi S)_A = \chi_{SA}$. The 2-category $\mathrm{Mon}[\mathcal{K},\mathcal{K}]$ so constituted is also called the *2-category* $2\text{-}\mathrm{Mnd}(\mathcal{K})$ *of 2-monads on* $\mathcal{K}$; we have referred to it several times in the Introduction. The action of $\mathcal{M}$ on $\mathcal{A}$ is here the evaluation 2-functor $e : [\mathcal{K},\mathcal{K}] \times \mathcal{K} \to \mathcal{K}$ given on objects by $e(T, A) = TA$; so that an action $a : TA \to A$ of a 2-monad T on $A \in \mathcal{K}$ has its usual classical meaning as above.

Suppose now that the 2-category $\mathcal{K}$ admits the *cotensor products* B^H where $B \in \mathcal{K}$ and H is a small category; since [18, Proposition 4.4] shows how to construct cotensor products from products, inserters, and equifiers, $\mathcal{K}$ surely admits the cotensor products above when it admits all (weighted) limits of these kinds, and in particular when it admits all *flexible* limits in the sense of [4], and certainly therefore when it is complete (as a 2-category). Then, for objects $A, B \in \mathcal{K}$, seen as 2-functors $A, B : 1 \to \mathcal{K}$, we have the (pointwise) right Kan extension $\langle A, B\rangle : \mathcal{K} \to \mathcal{K}$ of B along A; for by [16, Section 4.1], this is given on objects by $\langle A, B\rangle C = B^{\mathcal{K}(A,C)}$, its counit $\epsilon_{A,B} : \langle A, B\rangle A \to B$ being the evident morphism $B^{\mathcal{K}(A,A)} \to B^1 = B$. These $\langle A, B\rangle \in \mathcal{K}$ constitute the values on objects of a 2-functor $\langle -, -\rangle : \mathcal{K}^{op} \times \mathcal{K} \to [\mathcal{K},\mathcal{K}]$ which, by a basic property of Kan extensions, participates in a 2-natural isomorphism of categories

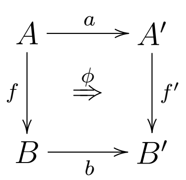
$$\pi_{T,A,B} : \mathcal{K}(TA, B) \cong [\mathcal{K},\mathcal{K}](T, \langle A, B\rangle),$$

exhibiting the 2-functor $\langle A, -\rangle : \mathcal{K} \to [\mathcal{K},\mathcal{K}]$ as a right adjoint of $e(-, A) : [\mathcal{K},\mathcal{K}] \to \mathcal{K}$. Accordingly we can apply the general theory above to conclude that:

2.1. Lemma. *When* $\mathcal{K}$ *admits cotensor products, for each* $A \in \mathcal{K}$ *the right Kan extension* $\langle A, A\rangle$ *of* A *along itself is a 2-monad on* $\mathcal{K}$*; moreover, for every 2-monad* T *on* $\mathcal{K}$*, there is an isomorphism of categories whose object part is a bijection between the monad morphisms* $\alpha : T \to \langle A, A\rangle$ *and the actions* $a : TA \to A$*, and whose morphism part is a bijection between the monad modifications* $\chi : \alpha_1 \to \alpha_2 : T \to \langle A, A\rangle$ *and those 2-cells* $\theta : a_1 \to a_2$ *for which* $(1, \theta) : (A, a_2) \to (A, a_1)$ *is a lax* T*-morphism. It follows that* χ *is invertible, or is an identity, precisely when* θ *is so.*

The content of the lemma goes back at least to Dubuc's thesis [9], wherein $\langle A, A\rangle$ is called the *codensity monad*; other authors have called it the *model-induced monad*; we may observe that it generalizes what in Lawvere's thesis [23] was called the *structure* of a functor A with codomain **Set**. The further applications of the ideas above, to which we now turn, go back to [14, Section 3]; we recall them because we need the details.

For the first of these we again take for $\mathcal{M}$ the monoidal category $[\mathcal{K},\mathcal{K}]$, but now we take for $\mathcal{A}$ the 2-category $\mathrm{Colax}[\mathbb{2},\mathcal{K}]$ of 2-functors $\mathbb{2} \to \mathcal{K}$, *colax* transformations between these, and modifications of the latter. Explicitly, an object of $\mathcal{A}$ is a morphism $f : A \to B$ in $\mathcal{K}$, a morphism $f \to f'$ in $\mathcal{A}$ is a triple (a, ϕ, b) giving a diagram



in $\mathcal{K}$, and a 2-cell $(a_1, \phi_1, b_1) \to (a_2, \phi_2, b_2) : f \to f'$ in $\mathcal{A}$ is a pair $(\xi : a_1 \to a_2, \eta : b_1 \to b_2)$ of 2-cells in $\mathcal{K}$ for which

$$\begin{array}{ccc} A & \overset{a_2}{\underset{a_1}{\genfrac{}{}{0pt}{}{\longrightarrow}{\longrightarrow}}}\ \xi\Uparrow & A' \\ {\scriptstyle f}\downarrow & \phi_1\Uparrow & \downarrow{\scriptstyle f'} \\ B & \underset{b_1}{\longrightarrow} & B' \end{array} \quad = \quad \begin{array}{ccc} A & \overset{a_2}{\longrightarrow} & A' \\ {\scriptstyle f}\downarrow & \phi_2\Uparrow & \downarrow{\scriptstyle f'} \\ B & \overset{b_2}{\underset{b_1}{\genfrac{}{}{0pt}{}{\longrightarrow}{\longrightarrow}}}\ \eta\Uparrow & B' \end{array} .$$

Here the action $\mathcal{M} \times \mathcal{A} \to \mathcal{A}$, or $[\mathcal{K}, \mathcal{K}] \times \mathrm{Colax}[\mathbb{2}, \mathcal{K}] \to \mathrm{Colax}[\mathbb{2}, \mathcal{K}]$, sends the object (T, f) to $Tf : TA \to TB$, and is thereafter most easily described by giving its partial 2-functors $T(-) : \mathrm{Colax}[\mathbb{2}, \mathcal{K}] \to \mathrm{Colax}[\mathbb{2}, \mathcal{K}]$ and $(-)f : [\mathcal{K}, \mathcal{K}] \to \mathrm{Colax}[\mathbb{2}, \mathcal{K}]$. The first of these sends $(a, \phi, b) : f \to f'$ to $(Ta, T\phi, Tb) : Tf \to Tf'$, and sends $(\xi, \eta) : (a_1, \phi_1, b_1) \to (a_2, \phi_2, b_2)$ to $(T\xi, T\eta) : (Ta_1, T\phi_1, Tb_1) \to (Ta_2, T\phi_2, Tb_2)$; the second sends $\alpha : T \to S$ to $(\alpha A, 1_{\alpha B.Tf}, \alpha B) : Tf \to Tf'$, and sends $\chi : \alpha \to \beta$ to $(\chi A, \chi B)$.

Now one easily verifies that $(a, \bar{f}, b) : Tf \to f$ is an action of the 2-monad T on $f : A \to B$ precisely when $a : TA \to A$ is an action of T on A and $b : TB \to B$ is an action of T on B, while $\bar{f}$ is a 2-cell $b.Tf \to f.a$ such that $(f, \bar{f})$ is a lax T-morphism $(A, a) \to (B, b)$. Moreover, if $(a_1, \bar{f}_1, b_1)$ and $(a_2, \bar{f}_2, b_2)$ are two actions of T on f, to give a 2-cell $\theta = (\xi, \eta) : (a_1, \bar{f}_1, b_1) \to (a_2, \bar{f}_2, b_2)$ for which $(1_f, \theta)$ is a lax T-morphism is clearly to give 2-cells $\xi : a_1 \to a_2$ and $\eta : b_1 \to b_2$, satisfying

$$\begin{array}{ccc} TA & \overset{a_2}{\underset{a_1}{\genfrac{}{}{0pt}{}{\longrightarrow}{\longrightarrow}}}\ \xi\Uparrow & A \\ {\scriptstyle Tf}\downarrow & \bar{f}_1\Uparrow & \downarrow{\scriptstyle f} \\ TB & \underset{b_1}{\longrightarrow} & B \end{array} \quad = \quad \begin{array}{ccc} TA & \overset{a_2}{\longrightarrow} & A \\ {\scriptstyle Tf}\downarrow & \bar{f}_2\Uparrow & \downarrow{\scriptstyle f} \\ TB & \overset{b_2}{\underset{b_1}{\genfrac{}{}{0pt}{}{\longrightarrow}{\longrightarrow}}}\ \eta\Uparrow & B \end{array} ,$$

for which $(1, \xi) : (A, a_2) \to (A, a_1)$ and $(1, \eta) : (B, b_2) \to (B, b_1)$ are lax T-morphisms.

Suppose now that the 2-category $\mathcal{K}$ admits products, inserters, and equifiers. Then as above we have the 2-functor $\langle -, - \rangle$ participating in the 2-adjunction $\pi : \mathcal{K}(TA, B) \cong [\mathcal{K}, \mathcal{K}](T, \langle A, B \rangle)$. Given morphisms $f : A \to B$ and $f' : A' \to B'$ in $\mathcal{K}$, the existence in $\mathcal{K}$ and hence in $[\mathcal{K}, \mathcal{K}]$ of products and inserters allows us to form the comma object $\{f, f'\}_l$ in

$$\begin{array}{ccccc} & & \langle A, A' \rangle & & \\ & \overset{\partial_0}{\nearrow} & & \overset{\langle A, f' \rangle}{\searrow} & \\ \{f, f'\}_l & & \lambda\Uparrow & & \langle A, B' \rangle \\ & \underset{\partial_1}{\searrow} & & \underset{\langle f, B' \rangle}{\nearrow} & \\ & & \langle B, B' \rangle & & \end{array} ,$$

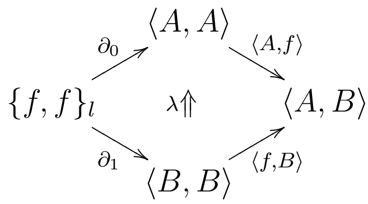
the subscript l (for lax) recalling that λ is not required to be invertible. Now to give a morphism $\gamma : T \to \{f, f'\}_l$ in $[\mathcal{K}, \mathcal{K}]$ is equally to give morphisms $\alpha : T \to \langle A, A' \rangle$ and

$\beta : T \to \langle B, B'\rangle$, along with a 2-cell $\psi : \langle f, B'\rangle\beta \to \langle A, f'\rangle\alpha$; whereupon we have $\alpha = \partial_0\gamma$, $\beta = \partial_1\gamma$, and $\psi = \lambda\gamma$. Moreover, for $\gamma_1, \gamma_2 : T \to \{f, f'\}_l$, to give a 2-cell $\chi : \gamma_1 \to \gamma_2$ is equally to give 2-cells $\rho : \alpha_1 \to \alpha_2$ and $\sigma : \beta_1 \to \beta_2$ for which $\langle A, f'\rangle\rho.\psi_1 = \psi_2.\langle f, B'\rangle\sigma$; whereupon $\rho = \partial_0\chi$ and $\sigma = \partial_1\chi$. To give α and β, however, is, as we saw, equally to give $a : TA \to A'$ and $b : TB \to B'$; whereupon to give $\psi : \langle f, B'\rangle\beta \to \langle A, f'\rangle\alpha$ is equally to give a 2-cell $\phi : b.Tf \to f'.a$. Again, to give ρ and σ is equally to give $\xi : a_1 \to a_2$ and $\eta : b_1 \to b_2$, while $\langle A, f'\rangle\rho.\psi_1 = \psi_2.\langle f, B'\rangle\sigma$ is equivalent to $(f'\xi)\phi_1 = \phi_2(\eta.Tf)$, which is the condition for (ξ, η) to be a 2-cell $(a_1, \phi_1, b_1) \to (a_2, \phi_2, b_2)$ in Colax$[\mathbf{2}, \mathcal{K}]$. Thus, for the given f and f' and for any T, we have an isomorphism of categories

$$\mathrm{Colax}[\mathbf{2}, \mathcal{K}](Tf, f') \cong [\mathcal{K}, \mathcal{K}](T, \{f, f'\}_l).$$

This isomorphism being 2-natural in T by the corresponding 2-naturality of the earlier isomorphism $\mathcal{K}(TA, B) \cong [\mathcal{K}, \mathcal{K}](T, \langle A, B\rangle)$, there is a unique extension of the object-values $\{f, f'\}_l$ to a 2-functor $\{-, -\}_l : (\mathrm{Colax}[\mathbf{2}, \mathcal{K}])^{op} \times \mathrm{Colax}[\mathbf{2}, \mathcal{K}] \to [\mathcal{K}, \mathcal{K}]$ rendering the present isomorphism 2-natural in each of the variables T, f, and f'.

So we are now in a position to apply the general considerations above to the case of the comma object

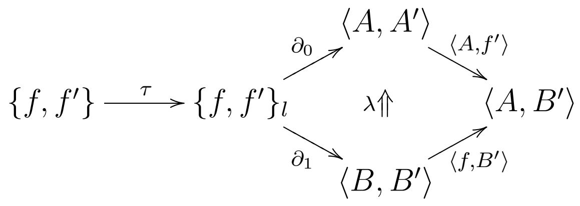


where $f = f'$, concluding that $\{f, f\}_l$ is a 2-monad on $\mathcal{K}$ and that monad morphisms $\gamma : T \to \{f, f\}_l$ correspond to actions $(a, \bar{f}, b) : Tf \to f$, with the corresponding result for monad modifications. However there is one new point that arises now, in that such a monad morphism γ corresponds not only to the action $(a, \bar{f}, b)$ but also to a third term: namely (α, ψ, β), where $\alpha : T \to \langle A, A\rangle$ and $\beta : T \to \langle B, B\rangle$ are the monad morphisms corresponding to the actions $a : TA \to A$ and $b : TB \to B$, while $\psi : \langle f, B\rangle\beta \to \langle A, f\rangle\alpha$ corresponds to $\bar{f}$. However these are also given by $\alpha = \partial_0\gamma$, $\beta = \partial_1\gamma$, and $\psi = \lambda\gamma$. It follows that $\partial_0\gamma$ and $\partial_1\gamma$ are monad morphisms whenever γ is a monad morphism; so that, taking $\gamma = 1$, we conclude that ∂_0 and ∂_1 are themselves monad morphisms. Putting this observation together with the general theory gives:

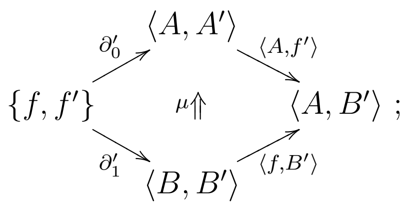
2.2. Lemma. *When $\mathcal{K}$ admits products, inserters, and equifiers, the comma object $\{f, f\}_l$ above is a 2-monad on $\mathcal{K}$, while $\partial_0 : \{f, f\}_l \to \langle A, A\rangle$ and $\partial_1 : \{f, f\}_l \to \langle B, B\rangle$ are monad morphisms. Moreover, for every 2-monad T on $\mathcal{K}$, there is an isomorphism of categories whose object part is a bijection between the monad morphisms $\gamma : T \to \{f, f\}_l$ and the triples $(a, \bar{f}, b)$ where $a : TA \to A$ and $b : TB \to B$ are actions of T for which $(f, \bar{f}) : (A, a) \to (B, b)$ is a lax T-morphism; here the monad morphism $\alpha : T \to \langle A, A\rangle$ corresponding in the sense of Lemma 2.1 to $a : TA \to A$ is $\partial_0\gamma$, and similarly the $\beta : T \to \langle B, B\rangle$ corresponding to $b : TB \to B$ is $\partial_1\gamma$. The morphism part of the isomorphism of categories is a bijection between monad modifications $\chi : \gamma_1 \to \gamma_2 : T \to \{f, f\}_l$ and pairs*

$(\xi : a_1 \to a_2, \eta : b_1 \to b_2)$ *for which* $(1,\xi) : (A,a_2) \to (A,a_1)$ *and* $(1,\eta) : (B,b_2) \to (B,b_1)$ *are lax* T*-morphisms satisfying* $(f,\bar{f}_1)(1,\xi) = (1,\eta)(f,\bar{f}_2)$*. Moreover* χ *is invertible or an identity precisely when this is true of both* ξ *and* η*.* ∎

Recall from [18] that a 2-category $\mathcal{K}$ admitting inserters and equifiers also admits inverters. Given $f : A \to B$ and $f' : A' \to B'$ therefore, and forming as above the comma object $\{f, f'\}_l$, we may consider the composite



wherein τ is the inverter of λ, and write this composite as



this diagram, of course, exhibits $\{f, f'\}$ as the *iso-comma-object*, and is universal among such diagrams in which the 2-cell, like μ, is invertible. Clearly we can imitate the proof of Lemma 2.2 with μ replacing λ, so that:

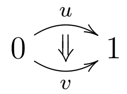
2.3. Lemma. *When* $\mathcal{K}$ *admits products, inserters, and equifiers, the results of Lemma 2.2 continue to hold when we replace* $\{f,f\}_l$ *by* $\{f,f\}$*, replace* ∂_0 *and* ∂_1 *by* ∂'_0 *and* ∂'_1*, and — in the second sentence of that lemma but not the third — replace lax* T*-morphism by* T*-morphism.* ∎

When $f' = f$, we have by this last lemma the 2-monad $\{f, f\}$, along with monad morphisms ∂'_0 and ∂'_1 corresponding to $\{f, f\}$-actions on A and on B, and μ corresponding to an enrichment of f to an $\{f, f\}$-morphism; whence it follows from Lemma 2.2 that $\tau : \{f, f\} \to \{f, f\}_l$ is a monad morphism. It is immediate that:

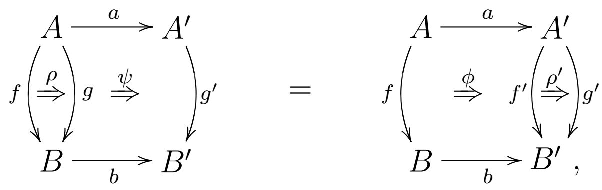
2.4. Lemma. *For a* $\mathcal{K}$ *admitting products, inserters, and equifiers, the monad morphism* $\gamma : T \to \{f,f\}_l$ *corresponding by Lemma 2.2 to the lax* T*-morphism* $(f,\bar{f}) : (A,a) \to (B,b)$ *factorizes through* τ *if and only if* $\bar{f}$ *is invertible.* ∎

2.5. Remark. As an inverter, τ is monomorphic and fully faithful in the 2-category $[\mathcal{K},\mathcal{K}]$. When $f' = f$, so that τ is a monad morphism, it is monomorphic in 2-Mnd$(\mathcal{K})$ = Mon$[\mathcal{K},\mathcal{K}]$ since it is so in $[\mathcal{K},\mathcal{K}]$. In fact it is also fully faithful in 2-Mnd$(\mathcal{K})$; but because we make no explicit use of this below, we omit the proof, which uses the characterization in Lemmas 2.2 and 2.3 of monad transformations.

Our final example of an action again has the monoidal 2-category $[\mathcal{K},\mathcal{K}]$ for $\mathcal{M}$, but now has for $\mathcal{A}$ the 2-category $\mathrm{Colax}[\mathbb{D},\mathcal{K}]$, where $\mathbb{D}$ denotes the 2-category



containing a single free 2-cell. So an object of $\mathcal{A}$ is a 2-cell $\rho : f \to g : A \to B$ in $\mathcal{K}$, a morphism $\rho \to \rho'$ is a quartet (a,ϕ,ψ,b) with



and a 2-cell $(a,\phi,\psi,b) \to (\bar{a},\bar{\phi},\bar{\psi},\bar{b})$ consists of 2-cells $\xi : a \to \bar{a}$ and $\eta : b \to \bar{b}$ satisfying the obvious condition. The monoidal 2-category $[\mathcal{K},\mathcal{K}]$ acts in an evident way on $\mathrm{Colax}[\mathbb{D},\mathcal{K}]$, the 2-functor $[\mathcal{K},\mathcal{K}] \times \mathrm{Colax}[\mathbb{D},\mathcal{K}] \to \mathrm{Colax}[\mathbb{D},\mathcal{K}]$ sending the object (T,ρ) to $T\rho : Tf \to Tg : TA \to TB$. To give an action $(a,\bar{f},\bar{g},b) : T\rho \to \rho$ of T on ρ is clearly to give actions $a : TA \to A$ and $b : TB \to B$, along with lax T-morphisms $(f,\bar{f})$ and $(g,\bar{g})$ from (A,a) to (B,b), for which ρ is a T-transformation.

If $\mathcal{K}$ is to admit products, inserters, and equifiers as before, and also to admit pullbacks, it must be complete: for it admits all conical limits and also cotensor products. When this is so, we can form for each $\rho : f \to g : A \to B$ in $\mathcal{K}$ the pullback

$$\begin{array}{ccc} [\rho,\rho']_l & \xrightarrow{\epsilon_0} & \{f,f'\}_l \\ {\scriptstyle \epsilon_1}\downarrow & & \downarrow{\scriptstyle \{f,\rho'\}_l} \\ \{g,g'\}_l & \xrightarrow[\{\rho,g'\}_l]{} & \{f,g'\}_l \end{array} ,$$

which participates in a 2-natural isomorphism

$$\mathrm{Colax}[\mathbb{D},\mathcal{K}](T\rho,\rho') \cong [\mathcal{K},\mathcal{K}](T,[\rho,\rho']_l).$$

Applying the general theory when $\rho' = \rho$ again gives an isomorphism of categories, connecting actions $T\rho \to \rho$ with monad morphisms $T \to [\rho,\rho]_l$; in fact we only use the object-part of this isomorphism.

In our applications, we wish to consider *given* T-algebras (A,a) and (B,b), and *given* lax T-morphisms $(f,\bar{f}),(g,\bar{g}) : (A,a) \to (B,b)$; and then to consider what *further* conditions T must satisfy if ρ is to be a T-transformation. Note that, since we have now supposed the 2-category $\mathcal{K}$ to be complete, the functor 2-category $[\mathcal{K},\mathcal{K}]$ admits all small limits, computed pointwise; whereupon, by a classical argument, the 2-category 2-Mnd$(\mathcal{K})$

of monoids in $[\mathcal{K},\mathcal{K}]$ admits all small limits, formed as in $[\mathcal{K},\mathcal{K}]$. Given $f, g : A \to B$, therefore, we can form in 2-Mnd($\mathcal{K}$) the diagram

$$\begin{array}{ccccc} & & \{f,f\}_l & \xrightarrow{\partial_0} & \langle A,A\rangle \\ & \nearrow^{\kappa_0} & & {\scriptstyle \partial_1}\searrow\!\!\!\!\!\nearrow{\scriptstyle \partial_0} & \\ (f|g)_l & & & & \\ & \searrow_{\kappa_1} & & & \\ & & \{g,g\}_l & \xrightarrow{\partial_1} & \langle B,B\rangle \ , \end{array}$$

where $\big((f|g)_l, \kappa_0, \kappa_1\big)$ is universal with the property that $\partial_0\kappa_0 = \partial_0\kappa_1$ and $\partial_1\kappa_0 = \partial_1\kappa_1$; in other words it is the equalizer of the evident pair of morphisms $\{f,f\}_l \times \{g,g\}_l \to \langle A,A\rangle \times \langle B,B\rangle$. Note that, when we take $\rho' = \rho$ in the pullback diagram above, we certainly have $\partial_0\epsilon_0 = \partial_0\epsilon_1$ and $\partial_1\epsilon_0 = \partial_1\epsilon_1$; thus we have a unique monad morphism $\nu : [\rho,\rho]_l \to (f|g)_l$ for which $\epsilon_0 = \kappa_0\nu$ and $\epsilon_1 = \kappa_1\nu$; and moreover ν is a strong monomorphim in 2-Mnd($\mathcal{K}$), since ϵ_0 and ϵ_1 constitute a jointly regular-monomorphic pair.

To give T-actions a on A and b on B, along with 2-cells $\bar{f}$ and $\bar{g}$ for which $(f,\bar{f})$ and $(g,\bar{g})$ are lax T-morphisms from (A,a) to (B,b), is by Lemma 2.2 to give monad morphisms $\gamma_0 : T \to \{f,f\}_l$ and $\gamma_1 : T \to \{g,g\}_l$ with $\partial_0\gamma_0 = \partial_0\gamma_1$ and $\partial_1\gamma_0 = \partial_1\gamma_1$; or equivalently to give a monad morphism $\delta : T \to (f|g)_l$. It is now clear from the considerations above that:

2.6. Lemma. *When $\mathcal{K}$ is complete, the object $[\rho,\rho]_l$ is a 2-monad, and we have a strongly-monomorphic monad morphism $\nu : [\rho,\rho]_l \to (f|g)_l$. For any 2-monad T, if lax T-morphisms $(f,\bar{f})$ and $(g,\bar{g})$ from (A,a) to (B,b) correspond as in Lemma 2.2 to monad morphisms $\gamma_0 : T \to \{f,f\}_l$ and $\gamma_1 : T \to \{g,g\}_l$, and hence to a single monad morphism $\delta : T \to (f|g)_l$, then $\rho : f \to g$ is a T-transformation if and only if the monad morphism δ factorizes (necessarily uniquely) through ν.* ■

2.7. Remark. We can similarly define $[\rho,\rho']$ as the pullback of $\{f,\rho'\} : \{f,f'\} \to \{f,g'\}$ and $\{\rho,g'\} : \{g,g'\} \to \{f,g'\}$, and define $(f|g)$ by replacing $\{f,f\}_l$ by $\{f,f\}$ and $\{g,g\}_l$ by $\{g,g\}$ in the limit-diagram defining $(f|g)_l$; clearly we have in 2-Mnd($\mathcal{K}$) a pullback

$$\begin{array}{ccc} [\rho,\rho] & \xrightarrow{\nu'} & (f|g) \\ \downarrow & & \downarrow \\ [\rho,\rho]_l & \xrightarrow[\nu]{} & (f|g)_l \ , \end{array}$$

and we have an analogue of Lemma 2.6 with ν' in place of ν.

There are various further such actions relevant to this paper, but we have chosen not to treat them in this general framework; we shall merely mention two here as further examples. In the first we once more take for $\mathcal{M}$ the monoidal 2-category $[\mathcal{K},\mathcal{K}]$, but now we take for $\mathcal{A}$ the functor 2-category $[\mathcal{C},\mathcal{K}]$, where $\mathcal{C}$ is an arbitrary 2-category; thus encompassing the "generalized algebras" borne by 2-functors $\mathcal{C} \to \mathcal{K}$. For the second we take for $\mathcal{M}$ the monoidal 2-category $\mathrm{Fin}[\mathcal{K},\mathcal{K}]$ of *finitary* endo-2-functors, 2-natural transformations, and modifications, with composition once more as its tensor product — see [5] — which acts on $\mathcal{K}$ by evaluation. A monoid in this $\mathcal{M}$ is just a finitary 2-monad on $\mathcal{K}$, and an action of such a monoid on an object of $\mathcal{K}$ is just an algebra structure for the 2-monad.

3. A general framework for the property-like and related conditions

For a 2-monad $T = (T, m, i)$ on a 2-category $\mathcal{K}$, we have met in the Introduction several conditions which fit into the following pattern. We suppose given T-algebras (A, a) and (B, b), and a morphism $f : A \to B$ in $\mathcal{K}$; and we impose upon T either the existence-and-uniqueness (E), or just the uniqueness (U), of a 2-cell $\bar{f}$ for which $(f, \bar{f}) : (A, a) \to (B, b)$ is either a lax T-morphism (L), a T-morphism (M), or a colax T-morphism (C); this condition being imposed for all T-algebras (A, a) and (B, b), and either (A) for *all* $f : A \to B$, or else (I) for all *invertible* $f : A \to B$. All told, we have here twelve conditions (XYZ), where X is A or I, where Y is E or U, and where Z is L, M, or C.

We can of course express the four conditions (XYL) in terms of the forgetful 2-functor $U_l : T\text{-}\mathrm{Alg}_l \to \mathcal{K}$ of the Introduction: write $(T\text{-}\mathrm{Alg}_l)_0$ and $\mathcal{K}_0$ for the ordinary categories underlying these 2-categories, and write $(U_l)_0 : (T\text{-}\mathrm{Alg}_l)_0 \to \mathcal{K}_0$ for the ordinary functor underlying the 2-functor U_l; then (AUL) is the assertion that $(U_l)_0$ is faithful, and (AEL) the assertion that it is fully faithful; while (IUL) may be expressed by saying that $(U_l)_0$ is *faithful on isomorphisms*, and (IEL) by saying that $(U_l)_0$ is *fully faithful on isomorphisms.* Similarly the conditions (XYM) and (XYC) can be expressed in terms of the forgetful functors $U_0 : T\text{-}\mathrm{Alg}_0 \to \mathcal{K}_0$ and $(U_c)_0 : (T\text{-}\mathrm{Alg}_c)_0 \to \mathcal{K}_0$.

3.1. Lemma. *For each of the six conditions of the form* (IYZ)*, it suffices to impose the condition not for all invertible $f : A \to B$ but only for the the special case where $B = A$ and f is the identity 1_A.*

Proof. The point is that, for an invertible f and T-actions $a : TA \to A$ and $b : TB \to B$, there is another T-action $a' : TA \to A$ given by $a' = f^{-1}.b.Tf$; and now f is an invertible strict T-morphism $(A, a') \to (B, b)$. Accordingly there is a bijection between, say, lax T-morphisms $(1_A, \theta) : (A, a) \to (A, a')$ and lax T-morphisms $(f, \bar{f}) : (A, a) \to (B, b)$, given by composition with f; whence the result is immediate. ∎

This lemma shows in particular that (IEM) is equivalent to what in the Introduction was called "essential uniqueness of T-algebra structure" (on an object of $\mathcal{K}$). The condition (AUM) is what was called "uniqueness of T-morphism structure"; and we took the conjunction (IEM) $\wedge$ (AUM) as our definition of "property-like", promising to prove it equivalent in Theorem 4.2 to "essential uniqueness of T-algebra structure on 2-functors of codomain $\mathcal{K}$". Further, the lax-idempotent 2-monads are by definition those satisfying (AEL), while the colax-idempotent 2-monads are those satisfying (AEC). Accordingly, with the goal of better understanding these conditions and related ones, we spend some time systematically analyzing the twelve conditions (XYZ) and their interconnections.

First we examine in this section a number of straightforward implications between them, which in later sections will be augmented by deeper connections needing some completeness properties of $\mathcal{K}$.

There are evident implications

$$(\mathrm{XEZ}) \Longrightarrow (\mathrm{XUZ})$$

for all X and Z, and

$$(\mathrm{AYZ}) \Longrightarrow (\mathrm{IYZ})$$

for all Y and Z, as well as

$$(\mathrm{XUL}) \Longrightarrow (\mathrm{XUM}) \Longleftarrow (\mathrm{XUC})$$

for all X.

Clearly the conjunction (IEM)$\wedge$(IUL) implies (IEL): for the "existence part" of (IEM) implies the "existence part" of (IEL), and (IUL) is precisely the "uniqueness part" of (IEL). In fact the converse is also true:

3.2. Proposition. *Condition* (IEL) *is equivalent to the conjunction* (IEM)$\wedge$(IUL)*; and dually with L replaced by C.*

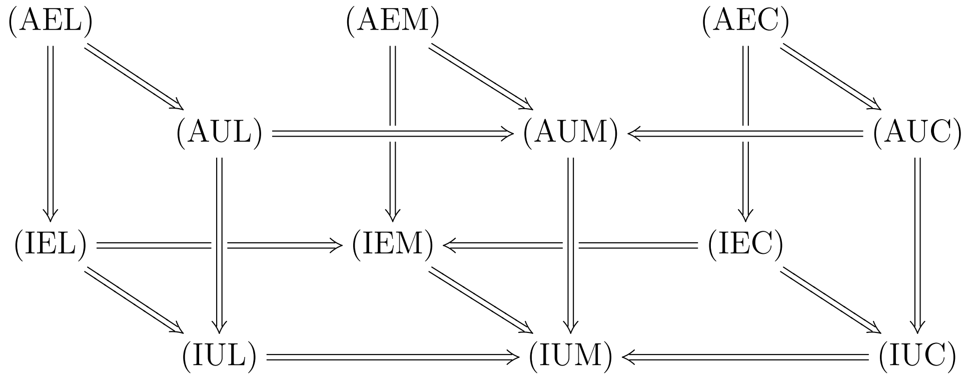
Proof. We have only to show that (IEL) implies (IEM) $\wedge$ (IUL); but trivially (IEL) implies (IUL), so that we need only prove that (IEL) implies (IEM). For the "uniqueness part" of (IEM) there is no problem, so it will suffice to show that every lax T-morphism $(f, \bar{f}) : (A, a) \to (B, b)$ with f invertible has $\bar{f}$ invertible. With $g = f^{-1}$, (IEL) gives us a lax T-morphism $(g, \bar{g}) : (B, b) \to (A, a)$. The composite lax T-morphism $(h, \bar{h}) = (g, \bar{g})(f, \bar{f}) : (A, a) \to (A, a)$ has $h = gf = 1_A$, so that $\bar{h}$ is an identity by (IEL), and $(g, \bar{g})(f, \bar{f})$ is an identity in T-Alg_l; similarly $(f, \bar{f})(g, \bar{g})$ is an identity, and so $(f, \bar{f})$ is invertible in T-Alg_l. But clearly a lax T-morphism $(f, \bar{f})$ is invertible if and only if both f and $\bar{f}$ are invertible, so that $(f, \bar{f})$ is indeed a T-morphism. ∎

Now for any 2-monad T on $\mathcal{K}$, a lax T-morphism $(f, \bar{f}) : (A, a) \to (B, b)$ with f invertible determines a canonical colax T-morphism $(f^{-1}, \tilde{f}) : (B, b) \to (A, a)$ where $\tilde{f}$ is the 2-cell $f^{-1}.\bar{f}.Tf^{-1} : f^{-1}.b = f^{-1}.b.Tf.Tf^{-1} \to f^{-1}.f.a.Tf^{-1} = a.Tf^{-1}$. Likewise a colax T-morphism structure on f determines a canonical lax T-morphism structure on f^{-1}, and these processes are mutually inverse, giving the equivalences:

$$(\mathrm{IUL}) \Longleftrightarrow (\mathrm{IUC})$$

$$(\mathrm{IEL}) \Longleftrightarrow (\mathrm{IEC}).$$

To sum up, then, we have a diagram of implications



wherein moreover the bottom squares are "pullbacks" by Proposition 3.2, and the conditions (IYL) and (IYC) are equivalent.

In our definition of the conditions (XYZ), the T-algebras (A, a) and (B, b) were given by actions $a : TA \to A$ and $b : TB \to B$ on objects $A, B \in \mathcal{K}$. As we have already remarked, however, one can equally consider "generalized T-algebras" (A, a), where A is a 2-functor $A : \mathcal{C} \to \mathcal{K}$ and $a : TA \to A$ is a 2-natural transformation. Then a lax T-morphism $(f, \bar{f}) : (A, a) \to (B, b)$ is given by a 2-natural transformation $f : A \to B$ and a modification $\bar{f}$ satisfying the usual axioms as in the Introduction, while a T-transformation $\alpha : (f, \bar{f}) \to (g, \bar{g})$ is given by a modification $\alpha : f \to g$ satisfying the usual axiom. Let us use $(\mathrm{XYZ})'$ for the condition like (XYZ), but imposed now for all *generalized* T-algebras as above. Of course

$$(\mathrm{XYZ})' \Longrightarrow (\mathrm{XYZ}),$$

since (XYZ) is just the case $\mathcal{C} = 1$ of $(\mathrm{XYZ})'$. Let us consider how far the converse of this is true. Given (A, a), (B, b) and $f : A \to B$ as above, A and B being 2-functors $\mathcal{C} \to \mathcal{K}$, we have for each $C \in \mathcal{C}$ the T-algebras (AC, aC) and (BC, bC); and to give $\bar{f}$ as above satisfying the axioms of the Introduction is to give for each C a lax T-morphism $(fC, \bar{f}C) : (AC, aC) \to (BC, bC)$, where these $\bar{f}C$ satisfy the *modification* condition, which requires for each $k : C \to D$ in $\mathcal{C}$ an equality

$$\begin{array}{ccc} TAC & \xrightarrow{aC} & AC \\ {\scriptstyle TfC}\downarrow & \bar{f}C \nearrow & \downarrow{\scriptstyle fC} \\ TBC & \xrightarrow[bC]{} & BC \\ {\scriptstyle TBk}\downarrow & & \downarrow{\scriptstyle Bk} \\ TBD & \xrightarrow[bD]{} & BD \end{array} \quad = \quad \begin{array}{ccc} TAC & \xrightarrow{aC} & AC \\ {\scriptstyle TAk}\downarrow & & \downarrow{\scriptstyle Ak} \\ TAD & \xrightarrow{aD} & AD \\ {\scriptstyle TfD}\downarrow & \bar{f}D \nearrow & \downarrow{\scriptstyle fD} \\ TBD & \xrightarrow[bD]{} & BD \, . \end{array}$$

3.3. Lemma. $(\mathrm{XYZ})'$ *coincides with* (XYZ) *when* Y=U, *and when* X=A.

Proof. When Y=U, we are concerned only with uniqueness, and the uniqueness of $\bar{f}$ follows from that of the $\bar{f}C$. When Y=E we need existence as well; but we have the existence of the $\bar{f}C$, and so need only the equality displayed in the diagram above. This, however, when X=A, follows from the uniqueness of $\bar{g}$ in $(g, \bar{g}) : (AC, aC) \to (BD, bD)$, where $g = Bk.fC = fD.Ak$. (When X=I this argument fails, since this g is not invertible in general.) ∎

So the only "primed" cases we need consider are three: (IEL)$'$, (IEM)$'$, and (IEC)$'$. Since the only obstruction to the truth of (IEZ)$'$, given (IEZ), is the equality in the diagram above, which deals with a single arrow $k : C \to D$, it is clear that (IEZ)$'$ holds for a general $\mathcal{C}$ if it does so for the two special cases $\mathcal{C} = 1$ (giving (IEZ)) and $\mathcal{C} = \mathbb{2}$, where this $\mathbb{2}$ is the arrow category $(0 \to 1)$, seen as a 2-category whose only 2-cells are identities. However we easily see , on taking A and B to be constant functors, that (IEZ)$'$ for the case $\mathcal{C} = \mathbb{2}$ contains (IEZ), so that:

3.4. Lemma. *T satisfies* (IEZ)$'$ *for a general 2-category $\mathcal{C}$ if it does so in the special case $\mathcal{C} = \mathbb{2}$.* ∎

It is clear that Lemma 3.1 applies equally to the case (IYZ)$'$ of generalized T-algebras. Combining that with the last lemma gives:

3.5. Lemma. (IEL)$'$ *is equivalent to the following condition, which we might call "*(IEL) *with naturality":* (IEL) *holds; and if a_1 and a_2 are T-actions on A and b_1 and b_2 are T-actions on B, and if $f : A \to B$ is both a strict T-morphism $(A, a_1) \to (B, b_1)$ and a strict T-morphism $(A, a_2) \to (B, b_2)$, then we have $f.\theta = \phi.Tf$ in $\mathcal{K}$, where $\theta : a_2 \to a_1$ and $\phi : b_2 \to b_1$ are the unique 2-cells, guaranteed by* (IEL)*, for which $(1, \theta) : (A, a_1) \to (B, b_1)$ and $(1, \phi) : (A, a_2) \to (B, b_2)$ are lax T-morphisms; in other words, we have the equality $(f, 1_{fa_2})(1_A, \theta) = (1_B, \phi)(f, 1_{fa_1})$ of lax T-morphisms. Similarly for* (IEM)$'$ *and* (IEC)$'$.
∎

3.6. Remark. Of course, when T satisfies (IEL)$'$, we have the property $(f, 1)(1, \theta) = (1, \phi)(f, 1)$ of Lemma 3.5 even when the a_i and b_i are actions (in the more general sense) on 2-functors $A, B : \mathcal{C} \to \mathcal{K}$; for this is just (IEL)$'$ with $\mathcal{C}$ replaced by $\mathbb{2} \times \mathcal{C}$. Similarly for (IEM)$'$ and (IEC)$'$.

4. Property-like 2-monads

We asserted in the Introduction that (IEM)$'$ is equivalent to the conjunction (IEM) $\wedge$ (AUM), and agreed to call a 2-monad T on $\mathcal{K}$ *property-like* if it satisfied the latter: that is, if T-algebra structure on an object of $\mathcal{K}$ is essentially unique, and T-morphism structure on a morphism of $\mathcal{K}$ is unique. In this section we establish the equivalence above, in fact giving many further conditions equivalent to being property-like.

In Section 2 we described various connections between actions and monad morphisms, mediated by Kan extensions and other limits in $\mathcal{K}$. We now also need the following connection, of a more general nature, first sketched in [21, Sections 3.5–3.6]; it suffices to state it precisely (in somewhat more detail than we in fact use), leaving the easy verification to the reader.

4.1. Lemma. *If $T = (T, m, i)$ and $S = (S, n, j)$ are 2-monads on $\mathcal{K}$, then the category* 2-Mnd$(\mathcal{K})(T, S)$ *of monad morphisms and monad modifications is isomorphic to the following category* Sact(T, S) *of "S-compatible actions of T on S": an object of* Sact(T, S) *is an action $u : TS \to S$, of T on the 2-functor S, for which $n : S^2 \to S$ is a strict T-morphism $(S^2, uS) \to (S, u)$; and a morphism $\theta : u \to v$ of* Sact(T, S) *is a modification $\theta : u \to v$ for which $(1_S, \theta)$ is a lax T-morphism $(S, v) \to (S, u)$ rendering commutative*

$$\begin{array}{ccc} (S^2, vS) & \xrightarrow{(1_{S^2}, \theta S)} & (S^2, uS) \\ {\scriptstyle (n, 1_{n.vS})} \downarrow & & \downarrow {\scriptstyle (n, 1_{n.uS})} \\ (S, v) & \xrightarrow[(1_S, \theta)]{} & (S, u) \end{array} .$$

The isomorphism Z : 2-Mnd$(\mathcal{K})(T, S) \to Sact(T, S)$ *sends $f : T \to S$ to the composite*

$$TS \xrightarrow{fS} SS \xrightarrow{n} S$$

and similarly sends $\chi : f \to g : T \to S$ to $n.f\chi$; its inverse sends $u : TS \to S$ to the composite $u.Tj : T \to S$ and similarly sends $\theta : u \to v$ to $\theta.Tj : u.Tj \to v.Tj$. ■

Following [6], we call a functor $U : \mathcal{A} \to \mathcal{C}$ *pseudomonic* if it is faithful and if, moreover, it is *full on isomorphisms*: the latter means that any invertible $h : UA \to UA'$ in $\mathcal{C}$ is Ug for some (necessarily unique) $g : A \to A'$ in $\mathcal{A}$, which by an easy argument must itself be invertible. This notion is representable, in the sense that a functor $U : \mathcal{A} \to \mathcal{C}$ is pseudomonic if and only if $[\mathcal{B}, U] : [\mathcal{B}, \mathcal{A}] \to [\mathcal{B}, \mathcal{C}]$ is so for every category $\mathcal{B}$; accordingly we define an arrow $U : A \to C$ in a 2-category $\mathcal{K}$ to be pseudomonic if the functor $\mathcal{K}(B, U) : \mathcal{K}(B, A) \to \mathcal{K}(B, C)$ is so for each $B \in \mathcal{K}$; and we then say that U is pseudoepic in $\mathcal{K}$ if it is pseudomonic in $\mathcal{K}^{op}$.

We shall write 1 for the identity 2-monad on $\mathcal{K}$; it is initial in the 2-category 2-Mnd$(\mathcal{K})$, since the "unit law" forces any monad morphism $1 \to T = (T, m, i)$ to be i, and any monad modification $i \to i : 1 \to T$ to be the identity.

Recall that we have used $\mathbb{2}$ above for the free category on the graph $(u : 0 \to 1)$, sometimes seen as a 2-category. Inverting u here gives the quotient $\mathbb{I} = (v : 0 \to 1)$, which some call "the free-living isomorphism"; besides the identities of 0 and 1, it has two morphisms v and v^{-1}. Finally let 2 denote the set $\{0, 1\}$, seen as a discrete category; note that we have inclusion functors $\iota : 2 \to \mathbb{I}$ and $\kappa : 2 \to \mathbb{2}$.

4.2. Theorem. *For a 2-monad T on a 2-category $\mathcal{K}$ admitting products, inserters, and equifiers, the following conditions — of which (viii) is property-likeness — are equivalent:*

(*i*) (IEL)$'$;

(*ii*) (IEM)$'$;

(*iii*) (IEC)$'$;

(*iv*) (IEL) *with naturality, in the sense of Lemma 3.5;*

(*v*) (IEM) *with naturality;*

(*vi*) (IEC) *with naturality;*

(*vii*) (IEL) $\wedge$ (AUL)*;*

(*viii*) (IEM) $\wedge$ (AUM)*;*

(*ix*) (IEC) $\wedge$ (AUC)*;*

(*x*) (IEM) $\wedge$ (AUL)*;*

(*xi*) (IEM) $\wedge$ (AUC)*;*

(*xii*) *for any 2-monad* $S = (S, n, j)$ *on* $\mathcal{K}$*, and any monad morphisms* $f : T \to S$ *and* $g : T \to S$*, there is exactly one invertible monad modification* $\chi : f \to g$*;*

(*xiii*) *for any 2-monad* $S = (S, m, j)$ *on* $\mathcal{K}$*, and any monad morphisms* $f : T \to S$ *and* $g : T \to S$*, there is exactly one monad modification* $\chi : f \to g$*;*

(*xiv*) $i : 1 \to T$ *is pseudoepic in* 2-Mnd($\mathcal{K}$).

If moreover the 2-category 2-Mnd($\mathcal{K}$) *admits the tensor products* $c \cdot T$ *when* c *is* 2, $\mathbb{2}$, *or* $\mathbb{I}$ *as above, then the following conditions are also equivalent to those appearing above:*

(*xv*) *the monad morphism* $\iota \cdot T : 2 \cdot T \to \mathbb{I} \cdot T$ *is invertible;*

(*xvi*) *the monad morphism* $\kappa \cdot T : 2 \cdot T \to \mathbb{2} \cdot T$ *is invertible.*

Proof. We may omit consideration of conditions (*iii*), (*vi*), (*ix*), and (*xi*), since they are the duals of (*i*), (*iv*), (*vii*), and (*x*) respectively, while (*ii*), (*v*), (*viii*), (*xii*), (*xiii*), and (*xiv*) are self-dual as regards the sense of 2-cells.

First we prove the equivalence of (*i*), (*iv*), (*vii*), (*x*), (*xii*), and (*xiii*).

(*i*) $\Rightarrow$ (*xiii*). The monad morphisms f and g give rise as in Lemma 4.1 to actions $u : TS \to S$ and $v : TS \to S$ such that $n : S^2 \to S$ is both a strict T-morphism $(S^2, uS) \to (S, u)$ and a strict T-morphism $(S^2, vS) \to (S, v)$. Now (IEL)$'$ with $\mathcal{C} = \mathcal{K}$ gives a unique modification $\theta : u \to v$ for which $(1_S, \theta) : (S, v) \to (S, u)$ is a lax T-morphism; whence also $(1_{S^2}, \theta S) : (S^2, vS) \to (S^2, uS)$ is a lax T-morphism. Moreover, by Remark 3.6, (IEL)$'$ further ensures that $(n, 1)(1, \theta S) = (1, \theta)(n, 1) : (S^2, vS) \to (S, uS)$; and so we conclude from Lemma 4.1 that there is a unique monad modification $\chi : f \to g$, namely $\theta.Tj$.

(*xiii*) $\Rightarrow$ (*xii*). Immediate.

(*xii*) $\Rightarrow$ (*x*). Taking S in (*xii*) to be $\langle A, A\rangle$ and applying Lemma 2.1 gives (IEM). As for (AUL), to show that there is at most one $\bar{f}$ for which $(f, \bar{f}) : (A, a) \to (B, b)$ is a lax T-morphism is by Lemma 2.2 to show that there is at most one monad morphism

$\gamma : T \to \{f, f\}_l$ with $\partial_0\gamma = \alpha$ and $\partial_1\gamma = \beta$. If γ_0 and γ_1 are two such, then (xii) with $S = \{f, f\}_l$ gives a unique invertible monad modification $\chi : \gamma_0 \to \gamma_1$. Now $\partial_0\chi$ is an invertible monad modification from α to α, and so by (xii) must be the identity; in the same way $\partial_1\chi$ is the identity. Now the "morphism part" of Lemma 2.2 ensures that χ is an identity, whence $\gamma_0 = \gamma_1$ as required.

$(x) \Rightarrow (vii)$. Immediate from Proposition 3.2.

$(vii) \Rightarrow (iv)$. In the notation of Lemma 3.5, $(f, 1)(1, \theta)$ and $(1, \phi)(f, 1)$ are both lax T-morphisms from $(A, a_1) \to (B, b_2)$, with the same underlying morphism $f : A \to B$; so they coincide by (AUL).

$(iv) \Rightarrow (i)$. Part of Lemma 3.5.

This completes the proof of the equivalence of (i), (iv), (vii), (x), (xii), and $(xiii)$. Next we prove the equivalence of (ii), (v), $(viii)$, (x), and (xii).

$(ii) \Rightarrow (xii)$. One repeats the proof of $(i) \Rightarrow (xiii)$, while noting that $\theta.Tj$ is invertible if θ is so.

$(xii) \Rightarrow (x)$. Proved above.

$(x) \Rightarrow (viii)$. Immediate since (AUL) implies (AUM).

$(viii) \Rightarrow (v)$. Proved just like $(vii) \Rightarrow (iv)$.

$(v) \Rightarrow (ii)$. Part of Lemma 3.5.

Finally (xiv) is equivalent to $(xiii)$ because 1 is initial in 2-Mnd($\mathcal{K}$), while (xv) and (xvi) are restatements of (xii) and $(xiii)$ in the presence of the given tensor products. ∎

4.3. Remark.

(a) Conditions (vii), $(viii)$, and (ix) say precisely that the forgetful functors $(U_l)_0$, U_0, and $(U_c)_0$ are pseudomonic.

(b) We could have added further equivalent conditions (IEL) ∧ (AUM), (IEC) ∧ (AUM), (IEL)∧(AUC), (IEC)∧(AUL), so that property-likeness is equivalent to any condition of the form (IEZ) ∧ (AUW). The equivalence of all these further conditions is a straightforward consequence of the theorem and the results of Section 3.

(c) Condition $(xiii)$ implies that the property-likes are closed in 2-Mnd($\mathcal{K}$) under arbitrary colimits (including weighted ones). Moreover it follows from (xiv) that S is property-like when T is so if $q : T \to S$ is pseudo-epimorphic in 2-Mnd($\mathcal{K}$); so that in particular the property-likes are closed in 2-Mnd($\mathcal{K}$) under co-fully-faithful quotients in the sense of the Introduction.

Before leaving this section, we record the following consequence of Theorem 4.2:

4.4. Corollary. *Under the hypotheses of Theorem 4.2, there are implications* (AEM) ⇒ (AEL) *and* (AEM) ⇒ (AEC).

Proof. In Section 3 we saw that (AEM) implied (IEM) and (AUM), which is to say that (AEM) implies property-likeness. So from the theorem above it follows that (AEM) implies (AUL). Now the "existence part" of (AEL) is already implied by the "existence part" of (AEM), while the "uniqueness part" of (AEL) is just (AUL). Thus (AEM) implies (AEL); and similarly (AEM) implies (AEC). ∎

5. Fully property-like 2-monads

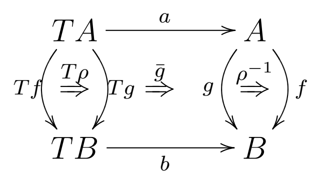
As we saw in the Introduction, a 2-monad T on $\mathcal{K}$ may or may not enjoy the property that every 2-cell between lax T-morphisms is a T-transformation; which is equally to say that the 2-functor $U_l : T\text{-Alg}_l \to \mathcal{K}$ is *full on 2-cells*, or *locally full* (in the sense that each functor $T\text{-Alg}_l((A,a),(B,b)) \to \mathcal{K}(A,B)$ is full); we now name this property (2L). Similarly we may say that T satisfies (2M) when $U : T\text{-Alg} \to \mathcal{K}$ is locally full, and that T satisfies (2C) when $U_c : T\text{-Alg}_c \to \mathcal{K}$ is locally full. Trivially we have

$$(2\text{L}) \Longrightarrow (2\text{M}) \Longleftarrow (2\text{C}).$$

The following proposition uses a transport of structure argument similar to that in Lemma 3.1.

5.1. Proposition. *Condition* (AUL) *is equivalent to the condition that $U_l : T\text{-}Alg_l \to \mathcal{K}$ be full on* invertible *2-cells; similarly* (AUM) *and* (AUC) *are just the conditions that U and U_c be full on invertible 2-cells. Thus* (2Z) *implies* (AUZ)*, for* Z=L, M*, or* C.

Proof. We prove that $U_l : T\text{-Alg}_l \to \mathcal{K}$ is full on invertible 2-cells if and only if T satisfies (AUL); the other results are similar. If $U_l : T\text{-Alg}_l \to \mathcal{K}$ is full on invertible 2-cells, and if $(f, \bar{f}_1) : (A,a) \to (B,b)$ and $(f, \bar{f}_2) : (A,a) \to (B,b)$ are lax T-morphisms, then by assumption the invertible 2-cell $1_f : f \to f$ must be a T-transformation $(f,\bar{f}_1) \to (f,\bar{f}_2)$, which means precisely that $\bar{f}_1 = \bar{f}_2$, giving (AUL). Conversely, if T satisfies (AUL), let $(f,\bar{f}) : (A,a) \to (B,b)$ and $(g,\bar{g}) : (A,a) \to (B,b)$ be lax T-morphisms and let $\rho : f \to g$ be an invertible 2-cell; then defining $\check{f}$ to be the composite



gives a lax T-morphism $(f,\check{f}) : (A,a) \to (B,b)$, so that by (AUL) we have $\bar{f} = \check{f}$; thus ρ is a T-transformation. ∎

It was noted in the Introduction that the 2-monad on **Cat** whose algebras are categories with finite coproducts satisfies condition (2L); however:

5.2. Proposition. *Even a property-like 2-monad may fail to satisfy* (2M)*;* a fortiori *it may fail to satisfy* (2L).

Proof. Write **Cart** for the 2-category of (small) categories with pullbacks, pullback-preserving functors, and cartesian natural transformations; by this last we mean those natural transformations for which the naturality squares are pullbacks. Write **2-Cat** for the 2-category of (small) 2-categories, 2-functors, and 2-natural transformations. We shall describe a 2-functor

$$\mathbf{Cart} \xrightarrow{(-)'} \mathbf{2\text{-}Cat}.$$

If $\mathcal{A}$ is a category with pullbacks, then $\mathcal{A}'$ is the 2-category whose underlying category is $\mathcal{A}$, and in which a 2-cell $\alpha : f \to g : A \to B$ is a subobject of A on which f and g coincide. Vertical composition of 2-cells is given by intersection of subobjects, while for α as above and 1-cells $h : B \to C$ and $k : D \to A$, the composite $h\alpha$ is the same subobject of A as α, while the composite αk is the pullback $k^*\alpha$. If now $\mathcal{C}$ is another category with pullbacks, and $F : \mathcal{A} \to \mathcal{C}$ preserves pullbacks, then F' is defined as the 2-functor with underlying functor F, which acts on 2-cells as F acts on the corresponding subobjects; of course F preserves monics since it preserves pullbacks. Finally the components $\phi A : FA \to GA$ of a cartesian natural transformation between pullback-preserving functors F and G constitute the components $\phi' A$ of a 2-natural transformation $\phi' : F' \to G'$. It is easy to verify that these constructions do indeed give a 2-functor from **Cart** to **2-Cat**, and as such take each monad (T, m, i) in **Cart** to a monad (T', m', i') in **2-Cat**. A T'-algebra is just a T-algebra and a T'-morphism is just a T-morphism — all T'-morphisms are strict, since $\mathcal{A}'$ has no non-identity invertible 2-cells. A T'-transformation between T'-morphisms f and g from (A, a) to (B, b) is a subobject $j : J \to A$ for which there is a pullback

$$\begin{array}{ccc} TJ & \xrightarrow{Tj} & TA \\ \downarrow & & \downarrow{\scriptstyle a} \\ J & \xrightarrow[j]{} & A. \end{array}$$

Now take $\mathcal{A}$ to be the category **Sgrp** of semigroups, and $T = (T, m, i)$ to be the monad on **Sgrp** whose algebras are the monoids, as in the penultimate paragraph of the Introduction. Since **Sgrp** has pullbacks preserved by T, while m and i are cartesian natural transformations (as was observed in [19, Section 5]), we have as above the 2-monad (T', m', i') on **Sgrp**$'$. This is property-like in the strong sense that T'-algebra structure on a given object is actually unique, as is T'-morphism structure. On the other hand, not all 2-cells are T'-transformations; for if A is any monoid, we have the 2-cell $\rho : 1_A \to 1_A : A \to A$ given by the subsemigroup $j : J \to A$, where j is the inclusion of the empty semigroup J; and this is not a T-transformation since there is no morphism from $TJ (= 1)$ to $J (= 0)$. Thus this T' is property-like without satisfying (2M). ∎

5.3. Definition. *We shall say that a 2-monad is* fully property-like *if it is property-like and moreover satisfies (2L) and (2C), and so also (2M).*

5.4. Remark. As foreshadowed in the Introduction, we shall prove in Section 6 below that the lax-idempotent 2-monads are fully property-like. We give no special name to the property-like 2-monads satisfying only (2M), since we know of no such that fails to satisfy (2L).

5.5. Remark. When $\mathcal{K}$ is complete, it follows from Lemma 2.6 that T satisfies (2L) if and only if, for each $\rho : f \to g : A \to B$ in $\mathcal{K}$, every monad morphism $\delta : T \to (f|g)_l$ factorizes through the strong monomorphism $\nu = \nu_\rho : [\rho, \rho]_l \to (f|g)_l$ of 2-Mnd($\mathcal{K}$). We conclude that, if $(\alpha_i : T_i \to S)$ is a jointly-epimorphic family in 2-Mnd($\mathcal{K}$) with each T_i satisfying (2L), then S satisfies (2L); in particular, the 2-monads satisfying (2L) are closed in 2-Mnd($\mathcal{K}$) under (conical) colimits and (merely epimorphic) quotients. Similarly, by duality, for the 2-monads satisfying (2C); and also, by Remark 2.7, for those satisfying (2M). Using Remark 4.3(c), we conclude that, for a complete $\mathcal{K}$, the fully property-like 2-monads are closed in 2-Mnd($\mathcal{K}$) under (conical) colimits and co-fully-faithful quotients.

The following result, although an immediate consequence of Proposition 5.1, is worth stating because many important structures (see [5]) are monadic not over **Cat** but only over $\mathbf{Cat}_g$, the full sub-2-category of **Cat** with the same objects and arrows, but in which all non-invertible 2-cells have been discarded.

5.6. Proposition. *If $\mathcal{K}$ is a 2-category in which every 2-cell is invertible, then for 2-monads on $\mathcal{K}$, the following conditions are equivalent:* (AUL), (AUM), (AUC), (2L), (2M), (2C)*; thus in particular every property-like 2-monad on such a $\mathcal{K}$ is fully property-like.* ∎

6. Lax-idempotent 2-monads

We establish in this section various properties of those 2-monads T on $\mathcal{K}$ that satisfy the condition (AEL); as we said in the Introduction, such a 2-monad will be said to be *lax-idempotent*, while a 2-monad satisfying (AEC) will be said to be *colax-idempotent*.

First recall from Section 3 that (AEL) $\Rightarrow$ (AUL) $\Rightarrow$ (AUM) and that (using Proposition 3.2) (AEL) $\Rightarrow$ (IEL) $\Rightarrow$ (IEM); that is to say:

6.1. Proposition. *Every lax-idempotent 2-monad (and dually every colax-idempotent 2-monad) is property-like.* ∎

Recall that, in any 2-category $\mathcal{K}$, we can speak of an *adjunction* $\eta, \epsilon : f \dashv u : A \to B$; here $u : A \to B$ and $f : B \to A$ are morphisms in $\mathcal{K}$, while the *unit* $\eta : 1 \to uf$ and the *counit* $\epsilon : fu \to 1$ are 2-cells satisfying the "triangular equations" $u\epsilon.\eta u = 1$ and $\epsilon f.f\eta = 1$; for the elementary theory of such adjunctions, see for instance [21]. We shall be concerned below with the special case of *an adjunction in $\mathcal{K}$ with identity counit*, obtained by requiring ϵ above to be an identity. To give such an adjunction $\eta, 1 : f \dashv u : A \to B$ is to give morphisms $u : A \to B$ and $f : B \to A$ with $fu = 1$, along with a 2-cell $\eta : 1 \to uf$ satisfying $\eta u = 1_u$ and $f\eta = 1_f$. Note that, given f and u with $fu = 1$, such an η is unique if it exists: for if we also have $\zeta : 1 \to uf$ with $\zeta u = 1$ and $f\zeta = 1$, the commutativity of

$$\begin{array}{ccc} 1 & \xrightarrow{\eta} & uf \\ {\scriptstyle \zeta}\downarrow & & \downarrow{\scriptstyle uf\zeta} \\ uf & \xrightarrow[\eta uf]{} & ufuf \end{array}$$

gives $\eta = \zeta$ since $uf\zeta$ and ηuf, like $f\zeta$ and ηu, are identities.

6.2. Theorem. *For a 2-monad $T = (T, m, i)$ on $\mathcal{K}$, the following are equivalent:*

(*i*) *T is lax-idempotent — that is, T satisfies* (AEL)*;*

(*ii*) *T satisfies* (AEL)$'$*;*

(*iii*) *in the 2-category* $[\mathcal{K}, \mathcal{K}]$*, there is a modification* $\lambda : 1 \to iT.m : T^2 \to T^2$ *giving an adjunction with identity counit* $\lambda, 1 : m \dashv iT : T \to T^2$*;*

(*iv*) *in the 2-category* $[\mathcal{K}, \mathcal{K}]$*, there is a modification* $\delta : Ti \to iT : T \to T^2$ *satisfying* $\delta i = 1$ *and* $m\delta = 1$*;*

(*v*) *for each* T*-algebra* (A, a) *there is a 2-cell* $\theta_{(A,a)} : 1 \to iA.a : TA \to TA$ *in* $\mathcal{K}$ *giving in* $\mathcal{K}$ *an adjunction with identity counit* $\theta_{(A,a)}, 1 : a \dashv iA : A \to TA$*.*

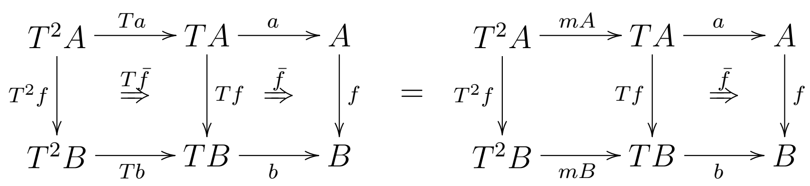
Proof. $(i) \Rightarrow (ii)$. By Lemma 3.3.

$(ii) \Rightarrow (iii)$. We have the T-algebras of domain $\mathcal{K}$ given by (T, m) and (T^2, mT), and we have the 2-natural $iT : T \to T^2$. By (AEL)$'$ there is a unique modification $\lambda : mT.TiT \to iT.m$ such that $(iT, \lambda) : (T, m) \to (T^2, mT)$ is a lax T-morphism. However $mT.TiT$, like $m.Ti$, is an identity; so that λ has the form $1 \to iT.m$. Of the two coherence conditions of the Introduction satisfied by the lax T-morphism (iT, λ), the first (or unit) condition is the assertion $\lambda.iT = 1$. Finally, the composite of the lax T-morphism $(iT, \lambda) : (T, m) \to (T^2, mT)$ and the strict T-morphism $(m, 1) : (T^2, mT) \to (T, m)$ has the form $(m.iT, m\lambda)$. Since $m.iT = 1$, the uniqueness assertion of (AEL)$'$ gives $m\lambda = 1$, completing the proof that we have an adjuntion $\lambda, 1 : m \dashv iT : T \to T^2$.

$(iii) \Rightarrow (iv)$. Given $\lambda, 1 : m \dashv iT : T \to T^2$, composing $\lambda : 1 \to iT.m : T^2 \to T^2$ with $Ti : T \to T^2$ gives a modification $\delta = \lambda.Ti : Ti \to iT.m.Ti = iT$. Moreover $m\delta = 1$ since $m\lambda = 1$; while $Ti.i = iT.i$ by the naturality of i, whence $\delta i = \lambda.Ti.i = \lambda.iT.i$, which is 1 since $\lambda.iT = 1$.

$(iv) \Rightarrow (v)$. Given a T-algebra (A, a), define a 2-cell $\theta_{(A,a)} : 1 \to iA.a : TA \to TA$ to be the composite of $Ta : T^2A \to TA$ with $\delta A : TiA \to iTA : TA \to T^2A$; since $Ta.TiA = 1$ because $a.iA = 1$, and since $Ta.iTA = iA.a$ by the naturality of i, the 2-cell $\theta_{(A,a)} = Ta.\delta A$ is indeed of the form $1 \to iA.a$. Moreover $\theta_{(A,a)}.iA = Ta.\delta A.iA$ is the identity 1 since $\delta i = 1$; while $a\theta_{(A,a)} = a.Ta.\delta A = a.mA.\delta A$ is the identity 1 since $m.\delta A = 1$.

$(v) \Rightarrow (i)$. For convenience, let us abbreviate the unit $\theta_{(A,a)}$ of the adjunction in (v) to θ_a. Consider T-algebras (A, a) and (B, b), together with a morphism $f : A \to B$ in $\mathcal{K}$. To give a 2-cell $\bar{f} : b.Tf \to f.a$ is, in view of the adjunction $\theta_a, 1 : a \dashv iA$, equally to give a 2-cell $\phi : b.Tf.iA \to f$, these being connected by the equations $\phi = \bar{f}.iA$ and $\bar{f} = (\phi a)(b.Tf.\theta_a)$. Since naturality of i gives $b.Tf.iA = b.iB.f = f$, the 2-cell ϕ has the form $f \to f$. If $(f, \bar{f})$ is to be a lax T-morphism from (A, a) to (B, b), the "unit" coherence condition requires $\bar{f}.iA = 1$; so we are forced to take $\phi = 1$, with $\bar{f} = b.Tf.\theta_a$ as the only possibility. It remains to show that this choice *does* give a lax T-morphism $(f, \bar{f}) : (A, a) \to (B, b)$; that is, we are to verify the remaining coherence condition



Since $a.mA = a.Ta$ and $b.mB = b.Tb$, this is a matter of proving equal the pair of 2-cells $\alpha, \beta : b.Tb.T^2 f \to f.a.Ta$, where α is the pasting composite on the left above and β that on the right.

However these correspond, under the adjunction $T\theta_a, 1 : Ta \dashv TiA : TA \to T^2 A$, to 2-cells $\bar\alpha, \bar\beta : b.Tb.T^2 f.TiA \to f.a$, where $\bar\alpha = \alpha.TiA$ and $\bar\beta = \beta.TiA$; so that it suffices to prove that $\bar\alpha = \bar\beta$. Since $Ta.TiA = 1$ and $T\bar f.TiA = T(\bar f.iA) = T1_f = 1_{Tf}$, the 2-cell $\bar\alpha$ reduces to $\bar f : b.Tf \to f.a$; and since $mA.TiA = 1$, the 2-cell $\bar\beta$ reduces to the same. This completes the proof. ∎

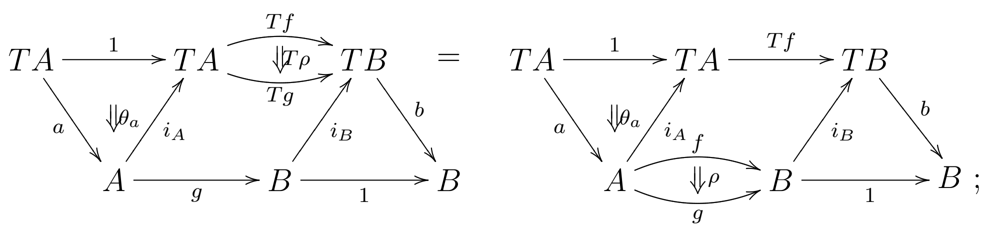
From the equivalence of (i) and (v) in this theorem, along with the proof above of $(v) \Rightarrow (i)$, we record for later reference the following:

6.3. Corollary. *If (A, a) is a T-algebra where T is lax-idempotent, we have in $\mathcal{K}$ an adjunction $\theta_a, 1 : a \dashv iA$; and if (B, b) is another T-algebra, the unique lax T-morphism $(f, \bar f) : (A, a) \to (B, b)$ has $\bar f = b.Tf.\theta_a$.*

We can now prove:

6.4. Proposition. (AEL) $\Longrightarrow$ (2L)

Proof. Let (A, a) and (B, b) be T-algebras where T satisfies (AEL), let $(f, \bar f), (g, \bar g) : (A, a) \to (B, b)$ be lax T-morphisms, and let $\rho : f \to g$ be any 2-cell in $\mathcal{K}$. Since $\bar f = b.Tf.\theta_a$ and $\bar g = b.Tg.\theta_a$, the condition for ρ to be a T-transformation reduces to the equality



which holds by 2-naturality of $i : 1 \to T$. ∎

Thus (AEL) implies (2L), and so for a lax-idempotent 2-monad we have not just that $(U_l)_0 : (T\text{-Alg}_l)_0 \to \mathcal{K}_0$ is fully faithful, but that $U_l : T\text{-Alg}_l \to \mathcal{K}$ is fully faithful as a 2-functor. Of course since (2L) implies (2M), so (AEL) implies (2M); on the other hand, to see that (AEL) implies (2C) we need first a result about the *colax* morphisms for a 2-monad satisfying (AEL).

6.5. Lemma. *If T is a lax-idempotent 2-monad and $(f, \tilde f) : (A, a) \to (B, b)$ is a colax T-morphism, then $\tilde f$ is invertible, and so $(f, \tilde f^{-1})$ is in fact a T-morphism.*

Proof. We show that $\tilde f$ is inverse to the $\bar f$ of Corollary 6.3. The composite $\bar f.iA$ is an identity because $(f, \bar f)$ is a lax T-morphism, while $\tilde f.iA$ is an identity because $(f, \tilde f)$ is a colax T-morphism; so that $(\bar f \tilde f).iA$ is an identity. Because of the adjunction $\theta_a, 1 : a \dashv iA$, however, composition with iA provides a bijection between 2-cells $\phi : f.a \to f.a$ and 2-cells $\phi.iA : f \to f$; so that $\bar f \tilde f = 1$ because $(\bar f \tilde f).iA = 1 = 1.iA$

It remains to show that $\tilde{f}\bar{f} : b.Tf \to b.Tf$ is an identity. However $b.Tf : (TA, mA) \to (B, b)$ is a strict T-morphism, so that $\tilde{f}\bar{f}$ is a T-transformation by Proposition 6.4. By the 2-dimensional aspect of the free-algebra property of TA, such a T-transformation is determined by its composite with iA; thus since $(\tilde{f}\bar{f}).iA$ is an identity as above, so too is $\tilde{f}\bar{f}$. ■

6.6. Corollary. (AEL) $\Longrightarrow$ (2C) ■

It now follows, using Corollary 4.4, that any of the conditions (AEZ) will imply each of (2L), (2M), and (2C). In particular, combining part of this with Proposition 6.1, we have:

6.7. Proposition. *Lax-idempotent 2-monads and colax-idempotent 2-monads are all fully property-like.* ■

6.8. Remark. In this section we have given a characterization of those 2-monads for which "structure is adjoint to unit" in terms of lax morphisms between *strict* algebras. Another characterization is obtained if we replace T-Alg_l by the 2-category Ps-T-Alg_l of *pseudo*-T-algebras, lax morphisms, and T-transformations, and ask that the forgetful functor to $\mathcal{K}$ be fully faithful.

Various authors have considered less strict notions of monad, and contemplated appropriate conditions involving an adjunction between structure and unit as here. In [22], Kock considered a 2-functor $T : \mathcal{K} \to \mathcal{K}$ and 2-natural transformations $m : T^2 \to T$ and $i : 1 \to T$ satisfying the unit conditions strictly, but with m being associative only up to coherent isomorphism; in [27], Street considered the general bicategorical notion of "monad" on a *bicategory*, calling them "doctrines"; in [25], Marmolejo considered (the formal theory of) *pseudo-monads*. All three authors had notions of a "monad with structure adjoint to unit" in a suitable sense, and in all three cases the monads in question could equally well have been characterized in terms of existence and uniqueness of lax-morphism structure.

The equivalence of (iii), (iv), and (v) in Theorem 6.2 has been proved in the less strict contexts mentioned above, as has Proposition 6.4. The equivalence of (i) and (ii) with the other conditions in Theorem 6.2 seems to be new, as do Lemma 6.5 and Corollary 6.6.

In Corollary 4.4 we saw that (AEM) implies (AEL) and (AEC), and thus their conjunction (AEL) $\wedge$ (AEC). We shall now show the converse.

6.9. Proposition. *For a 2-monad $T = (T, m, i)$, the following are equivalent:*

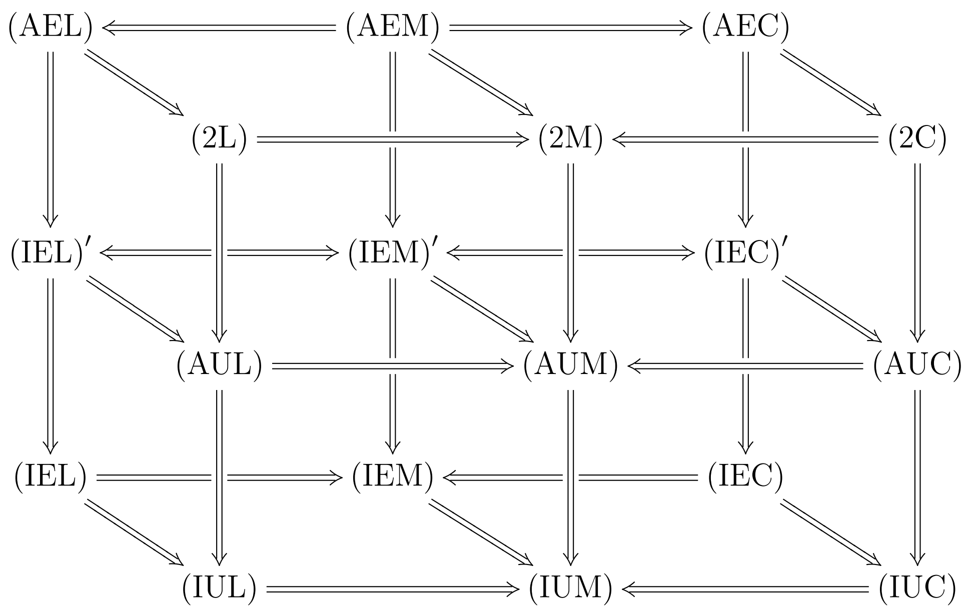
(i) *T satisfies* (AEM) *(in which case one might call T* pseudo-idempotent*);*

(ii) *there is an isomorphism $\lambda : iT.m \cong 1$ giving an adjoint equivalence $(\lambda, 1 : m \dashv iT) : T \simeq T^2$;*

(iii) (AEL) $\wedge$ (AEC).

Proof. One sees that (*ii*) follows from (*i*) by observing that if T satisfies not just (AEL) but (AEM) then the λ of Theorem 6.2(*iii*) is invertible. That (*ii*) implies (*iii*) follows from Theorem 6.2. Thus it remains only to show that (*iii*) implies (*i*). But by Lemma 6.5, if T satisfies (AEL) then any colax T-morphism is in fact a T-morphism, and the result follows. ∎

We are now ready to form an expanded version of the diagram of implications given at the end of Section 3, assuming now that the 2-category $\mathcal{K}$ has products, inserters, and equifiers. We include in it the conditions (2Z) involving T-transformations, and also those "primed" conditions which do not coincide with their "unprimed" counterparts:



Also proved, but not appearing in the diagram, are the equivalences:

$$\text{(IYL)} \iff \text{(IYC)}$$

$$\text{(IEM)} \wedge \text{(IUZ)} \iff \text{(IEZ)}$$

$$\text{(IEZ)} \wedge \text{(AUW)} \iff \text{(IEV)}'$$

$$\text{(AEL)} \wedge \text{(AEC)} \iff \text{(AEM)}.$$

6.10. Remark. In the case of a 2-category $\mathcal{K}$ with no non-invertible 2-cells, we have (AUM) $\iff$ (2M) by Proposition 5.6, and these conditions reduce to only five: (AEM), (IEM)′, (AUM), (IEM), and (IUM).

7. The case of ordinary monads

Among the 2-categories are those for which the only 2-cells are identities; such 2-categories are called *locally discrete*, and one usually identifies these locally-discrete 2-categories with mere categories. To give a 2-monad on a locally-discrete 2-category is just to give a monad on the corresponding category; and thus all that has gone before can be applied to ordinary monads by considering the case where $\mathcal{K}$ is locally discrete. For the rest of this section, therefore, $\mathcal{K}$ will be an ordinary category, viewed as a locally-discrete 2-category.

Because there are no non-identity 2-cells in $\mathcal{K}$, the 2-categories of algebras T-Alg$_l$, T-Alg, and T-Alg$_c$ for a monad T all coincide with T-Alg$_s$, which is just the Eilenberg-Moore category $\mathcal{K}^T$ for the monad T. It follows that, for each X and Y, the conditions (XYL), (XYM), and (XYC) are equivalent. Because the forgetful functor $U : T\text{-Alg} \to \mathcal{K}$ is faithful, the conditions (XUZ) are all vacuous; of course the conditions (2Z) are vacuous, and so we are left with only two distinct conditions, (IEM) and (AEM), which we now investigate.

The condition that an ordinary monad be property-like is just (IEM), since (AUM) is vacuous; interpreting Lemma 3.1 in the current context of a locally-discrete $\mathcal{K}$, we see that the ordinary monad T is property-like if and only if there is at most one T-action $a : TA \to A$ on any given object A of $\mathcal{K}$. To interpret Theorem 4.2 we first note that a locally-discrete 2-category is flexibly complete if and only if the corresponding category is complete.

7.1. Theorem. *For a monad T on a complete category $\mathcal{K}$, the following conditions are equivalent:*

(*i*) *T is property-like;*

(*ii*) *$U : T\text{-}Alg \to \mathcal{K}$ is injective on objects;*

(*iii*) *$U : T\text{-}Alg \to \mathcal{K}$ is pseudomonic;*

(*iv*) *$i : 1 \to T$ is an epimorphism in* $\mathrm{Mnd}(\mathcal{K})$ *;*

(*v*) *the identities $1 : T \to T$ and $1 : T \to T$ exhibit T as the coproduct $T+T$ in* $\mathrm{Mnd}(\mathcal{K})$*;*

(*vi*) *if S is any monad on $\mathcal{K}$, then there is at most one monad morphism from T to S;*

(*vii*) *if $A : \mathcal{C} \to \mathcal{K}$ is a functor with arbitrary domain, then there is at most one T-action on A.*

■

We now turn to the monads which satisfy (AEM), equivalent to (AEL) and (AEC). As the only adjunctions in an ordinary category are isomorphisms, it follows from Theorem 6.2 that a lax-idempotent 2-monad on a locally-discrete 2-category is just an idempotent monad on the corresponding ordinary category.

7.2. Proposition. *The following conditions on a monad (T, m, i) are equivalent:*

(i) *T is idempotent, meaning that m is invertible;*

(ii) *iT is invertible;*

(iii) *Ti is invertible;*

(iv) *m and iT are mutually inverse;*

(v) *m and Ti are mutually inverse;*

(vi) *$Ti = iT$;*

(vii) *$a : TA \to A$ is a T-action precisely when it is inverse to i_A;*

$(viii)$ *$U : T\text{-}Alg \to \mathcal{K}$ is full;*

(ix) *$U : T\text{-}Alg \to \mathcal{K}$ is fully faithful;*

(x) *for all natural transformations $f : A \to B : \mathcal{C} \to \mathcal{K}$ and all T-actions $a : TA \to A$ and $b : TB \to B$, the transformation f is a T-morphism from (A, a) to (B, b).*

Proof. Most of these equivalences are immediate consequences of Theorem 6.2; but in any case they are well known, except perhaps for $(i) \Leftrightarrow (x)$ — which is just the fact that (AEM) $\Leftrightarrow$ (AEM)$'$. ∎

7.3. Remark. Observe that the natural analogue in the "lax situation" of each of the conditions (iv), (vi), (ix), and (x) is a condition on a 2-monad equivalent to its being lax-idempotent, and appearing in Theorem 6.2; thus (iv) above corresponds to (iii) in Theorem 6.2, while (vi), (ix), and (x) above correspond, respectively, to (iv), (i), and (ii) in Theorem 6.2. Condition (vii) above corresponds to a slight modification of condition (v) of Theorem 6.2, involving *pseudo* T-algebras rather than strict ones.

7.4. Remark. As pointed out in the Introduction, every idempotent monad is of course property-like, but the converse is false, $U : \mathbf{Mon} \to \mathbf{Sgrp}$ providing a counter-example.

When $\mathcal{K} = \mathbf{Set}$ however, the two conditions are equivalent.

7.5. Proposition. *If the monad (T, m, i) on* **Set** *is property-like then it is idempotent.*

Proof. Let A and B be sets bearing T-actions and $f : A \to B$ a function. We shall show that f is a T-morphism. We may suppose A to be non-empty, for otherwise the result is trivial.

First suppose that f is a monomorphism (which of course we can think of as being a subset inclusion) and that the complement of A in B is not a single point. Then f is an equalizer in **Set** of the identity 1_B and some bijection $k : B \to B$. Now as T is property-like, $U : T\text{-Alg} \to \mathcal{K}$ is pseudomonic, and so both 1_B and k are T-morphisms, and so f too is a T-morphism, since U creates equalizers.

If f is a monomorphism, and the complement of the non-empty A is a single point, we may factorize f as $\binom{f}{f} : A \to B \times B$ followed by a projection $\pi : B \times B \to B$. Now $B \times B$ bears the product T-action, and π is a T-morphism; whence f will be a T-morphism if and only if $\binom{f}{f}$ is one. But now $\binom{f}{f}$ is monic, and the image has complement not equal to 1; and so we can apply the argument given above.

Finally suppose f is an arbitrary function $A \to B$. Factorize f as the monomorphism $\binom{1_A}{f} : A \to A \times B$ followed by the projection $\pi_B : A \times B \to B$. Once again $A \times B$ bears the product T-action and π_B is a T-morphism, while $\binom{1_A}{f}$, being monic, is a T-morphism by the previous two paragraphs; so f is itself a T-morphism.

Thus U is full, and so T is idempotent. ∎

In fact the idempotent monads on **Set** are of limited interest, insofar as (to within isomorphism) they are only three: the identity monad, the constant monad at 1, and the monad whose algebras are 0 and 1.

8. On the coreflectiveness of certain classes of monads and 2-monads

Let $\mathcal{F}$ be a family of morphisms in a category $\mathcal{A}$, which we suppose for simplicity to be finitely complete. An object X of $\mathcal{A}$ is said to be *coorthogonal* to $\mathcal{F}$ if $\mathcal{A}(X, f) : \mathcal{A}(X, M_f) \to \mathcal{A}(X, N_f)$ is invertible for each $f : M_f \to N_f$ in $\mathcal{F}$; and we shall further say that X is *weakly coorthogonal* to $\mathcal{F}$ if each $\mathcal{A}(X, f)$ is monomorphic. Clearly the full subcategory $\mathcal{F}^{\perp}$ of $\mathcal{A}$ given by the objects coorthogonal to $\mathcal{F}$ is closed in $\mathcal{A}$ under any colimits that exist in $\mathcal{A}$, while the full subcategory $\mathcal{F}^{\flat}$ given by the objects weakly coorthogonal to $\mathcal{F}$ is closed in $\mathcal{A}$ not only under colimits but "under all jointly-epimorphic families": by which we mean that, if the family $\mathcal{G} = (g : X_g \to Y)$ is jointly epimorphic with each $X_g \in \mathcal{F}^{\flat}$, then $Y \in \mathcal{F}^{\flat}$. Equally clearly, the full subcategory $\mathcal{F}^{\perp}$ is *itself* closed under jointly-epimorphic families if each $f \in \mathcal{F}$ is a strong monomorphism; while if each $f \in \mathcal{F}$ is merely monomorphic, $\mathcal{F}^{\perp}$ is closed under those families $(g : X_g \to Y)$ that are *jointly strongly-epimorphic*: a notion equivalent, in the presence of finite limits, to being *jointly extremal-epimorphic* in the sense that there is no proper subobject of Y through which every g factorizes.

Although the following results are so well known as to be essentially folklore, it is not easy to point to these precise statements in print:

8.1. Lemma. *Let $\mathcal{A}$ be a category with finite limits. (a) If $\mathcal{A}$ admits all intersections — even large ones, if need be — of strong subobjects, then a full subcategory $\mathcal{B}$ is coreflective with each counit $\epsilon A : PA \to A$ a strong monomorphism if and only if it is closed under jointly-epimorphic families. (b) If $\mathcal{A}$ admits arbitrary intersections of subobjects, then a full subcategory $\mathcal{B}$ is coreflective with each counit $\epsilon A : PA \to A$ a monomorphism if and only if it is closed under jointly strongly-epimorphic families.*

Proof. We can largely consider the two cases together. For the "only if" part, consider a family $\mathcal{G} = (g : X_g \to Y)$ with each $X_g \in \mathcal{B}$; then g is uniquely of the form $\epsilon Y.h_g$, where $h_g : X_g \to PY$. If $\mathcal{G}$ is jointly epimorphic [resp. jointly strongly-epimorphic], then

ϵY is epimorphic [resp. strongly-epimorphic], and hence is invertible, since it is strongly monomorphic [resp. monomorphic]; thus $Y \in \mathcal{B}$. As for the "if" part, consider for any given $A \in \mathcal{A}$ the family $\mathcal{G} = (g : B_g \to A)$ of all morphisms g with codomain A and with domain B_g lying in $\mathcal{B}$. Let $\epsilon A : PA \to A$ be the intersection of all the strong subobjects [resp. subobjects] of A through which each $g \in \mathcal{G}$ factorizes, say as $g = iA.h_g$. Then the family $\mathcal{H} = (h_g)_{g\in\mathcal{G}}$ is jointly epimorphic; for if $x, y : PA \to C$ satisfy $xh_g = yh_g$ for all $g \in \mathcal{G}$, then every h_g factorizes through the equalizer $z : D \to PA$ of x and y, which by the definition of PA must be invertible, giving $x = y$. Moreover, in case (b), the family $\mathcal{H}$ is jointly strongly-epimorphic, since it factorizes through no proper subobject of PA. In both cases, therefore, we have $PA \in \mathcal{B}$, and clearly $\epsilon A : PA \to A$ is the coreflection. ■

When the category $\mathcal{A}$ in the considerations above is replaced by a 2-category, we are concerned with the coreflectiveness of $\mathcal{B}$ not merely as a category but as a 2-category: for $B \in \mathcal{B}$ we want ϵA to induce an isomorphism $\mathcal{B}(B, PA) \cong \mathcal{A}(B, A)$ of categories, and not just an isomorphism $\mathcal{B}_0(B, PA) \cong \mathcal{A}_0(B, A)$ of sets. In fact one can give a 2-categorical version of everything above, wherein coorthogonality now requires an isomorphism $\mathcal{A}(X, f) : \mathcal{A}(X, M_f) \to \mathcal{A}(X, N_f)$ of *categories*, the meaning of "jointly-epimorphic family" is extended to include a 2-cell clause, and the concept of strong monomorphism is understood in a 2-categorical sense. It is not, however, worth our while here to develop these 2-categorical extensions in the abstract: for in our applications it is simple to give an *ad hoc* argument justifying the isomorphism $\mathcal{B}(B, PA) \cong \mathcal{A}(B, A)$, after having established (by Lemma 8.1 or otherwise) the simpler merely-categorical bijection $\mathcal{B}_0(B, PA) \cong \mathcal{A}_0(B, A)$.

There is one point, however, that is worth making here. A morphism $f : A \to B$ in a 2-category $\mathcal{A}$ should be called "monomorphic in $\mathcal{A}$" if each $\mathcal{A}(C, f) : \mathcal{A}(C, A) \to \mathcal{A}(C, B)$ is a monomorphism *of categories*; that is, if composition with f is injective both for morphisms and for 2-cells. This certainly *implies* that f is monomorphic in the underlying category $\mathcal{A}_0$, but in general it is stronger. Since, however, $\mathcal{A}(C, f)$ is monomorphic in **Cat** if and only if $\mathbf{Cat}_0(\mathbb{2}, \mathcal{A}(C, f))$ is monomorphic in **Set**, and since this is $\mathcal{A}_0(\mathbb{2} \cdot C, f)$ if the tensor product $\mathbb{2} \cdot C$ exists, it follows that the two senses of "monomorphism" coincide if $\mathcal{A}$ admits the tensor products $\mathbb{2} \cdot C$. They also coincide — and this is the case we need below — when the 2-category $\mathcal{A}$ admits pullbacks; for f is monomorphic in $\mathcal{A}$ [resp. in $\mathcal{A}_0$] if and only if the square

$$\begin{array}{ccc} A & \xrightarrow{1} & A \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

is a pullback in $\mathcal{A}$ [resp. in $\mathcal{A}_0$], and these last statements coincide if pullbacks in $\mathcal{A}$ exist.

We intend to apply the results above when $\mathcal{A}$ is the ordinary category $(\text{2-Mnd}(\mathcal{K}))_0$ for a 2-category $\mathcal{K}$. Observe first that the 2-category $[\mathcal{K}, \mathcal{K}]$ admits all limits — including weighted ones — that exist in $\mathcal{K}$, these being formed pointwise. We shall suppose henceforth that $\mathcal{K}$ admits all finite limits (in the sense of [17]); recall that it therefore admits inserters and equifiers, as well as the cotensor product $A^{\mathbb{2}}$ for $A \in \mathcal{K}$. So $[\mathcal{K}, \mathcal{K}]$

too is finitely complete in this sense; and in particular $f : T \to S$ is monomorphic in $[\mathcal{K}, \mathcal{K}]$ — the meaning of "monomorphic" now being unambiguous — if and only if each $fA : TA \to SA$ is monomorphic in $\mathcal{K}$. Next, it is easy to see that the forgetful 2-functor V : 2-Mnd$(\mathcal{K})$ = Mon$[\mathcal{K}, \mathcal{K}] \to [\mathcal{K}, \mathcal{K}]$ creates whatever limits exist in $[\mathcal{K}, \mathcal{K}]$ — again including weighted ones. So 2-Mnd$(\mathcal{K})$ is finitely complete, and in particular V preserves and reflects monomorphisms. Since, however, there is no simple description of the *strong* monomorphisms in $(\text{2-Mnd}(\mathcal{K}))_0$, we shall suppose in our applications of Lemma 8.1 that $\mathcal{K}$ *admits all intersections of subobjects*; so that $(\text{2-Mnd}(\mathcal{K}))_0$ does so too by the above, and *a fortiori* admits all intersections of *strong* subobjects.

Let us begin with the simple case of ordinary monads T on an ordinary category $\mathcal{K}$, where we have the following results involving direct applications of Lemma 8.1. In this case, of course, 2-Mnd$(\mathcal{K})$ reduces to the mere category Mnd$(\mathcal{K})$ of monads and monad morphisms — there being no 2-cells but identities.

8.2. Theorem. *Let the category $\mathcal{K}$ be complete and admit all intersections of subobjects. Then the full subcategory of property-like monads is coreflective in* Mnd$(\mathcal{K})$*, as is the full subcategory of idempotent monads. In each case, the counit of the coreflection is a strong monomorphism.*

Proof. We saw in Theorem 7.1 that T is property-like precisely when $i : 1 \to T$ is epimorphic in Mnd$(\mathcal{K})$; since such T are clearly closed in Mnd$(\mathcal{K})$ under jointly-epimorphic families, the first result follows from either (a) or (b) of Lemma 8.1 — but using (a) gives the result on the counit. As for idempotent monads, we have since $\mathcal{K}$ is complete the existence of $\langle A, A\rangle$, and the iso-comma object $\{f, f\}$ here reduces to a pullback

$$\begin{array}{ccccc} & & \langle A, A\rangle & & \\ & \overset{\partial'_0}{\nearrow} & & \overset{\langle A,f\rangle}{\searrow} & \\ \{f,f\} & & & & \langle A, B\rangle \\ & \underset{\partial'_1}{\searrow} & & \underset{\langle f,B\rangle}{\nearrow} & \\ & & \langle B, B\rangle & & \end{array} .$$

Clearly T is idempotent if and only if it is, for each $f : A \to B$ in $\mathcal{K}$, coorthogonal to the strong monomorphism $\phi_f : \{f, f\} \to \langle A, A\rangle \times \langle B, B\rangle$ having for its components ∂'_0 and ∂'_1. Since such T are closed in Mnd$(\mathcal{K})$ under jointly-epimorphic families, the result again follows by Lemma 8.1(a). ■

8.3. Remark. That the idempotent monads here are coreflective is well known, having first been proved by Fakir [12]. Although the argument above needs completeness of $\mathcal{K}$ to construct the $\langle A, A\rangle$, another proof may be given using the results of Day [8], which needs only *finite* completeness along with all intersections of *strong* monomorphisms.

We return now to our more general situation of 2-monads on a 2-category $\mathcal{K}$, and suppose now that $\mathcal{K}$ is complete. In the notation of Section 2, to say that T satisfies (AEL) is to say that, for each morphism $f : A \to B$ of $\mathcal{K}$, the 2-monad T is coorthogonal in $(\text{2-Mnd}(\mathcal{K}))_0$ to the morphism $\sigma_f : \{f, f\}_l \to \langle A, A\rangle \times \langle B, B\rangle$ whose components are ∂_0

and ∂_1. Similarly, to say that T satisfies (IEL) is to say that it is coorthogonal to σ_f for each *invertible* f; to say that T satisfies (AUL) is to say that it is weakly coorthogonal to each σ_f; and to say that T satisfies (IUL) is to say that it is weakly coorthogonal to those σ_f with f invertible. There are precisely analogous results wherein L is replaced by M and $\{f,f\}_l$ by $\{f,f\}$; and still others with L replaced by C. Moreover, we saw in Remark 5.5 that T satisfies (2L) if and only if it is coorthogonal to the strong monomorphism ν_ρ for each $\rho : f \to g : A \to B$ in $\mathcal{K}$, with similar results for (2M) and (2C). It follows that the 2-monads T satisfying any one or more of our twelve conditions (XYZ) and our three conditions (2Z) are closed in $(2\text{-Mnd}(\mathcal{K}))_0$ under colimits; while those T satisfying one or more of the six conditions (XUZ) and the three conditions (2Z) are moreover closed in $(2\text{-Mnd}(\mathcal{K}))_0$ under jointly-epimorphic families.

It is in fact the case that any 2-monad T satisfying one of our fifteen conditions "does so in a 2-categorical sense". We illustrate by the case (IEL), leaving the other cases to the reader. Let T then have the property that, for each isomorphism $f : A \to B$ in $\mathcal{K}$, each monad morphism $\delta : T \to \langle A,A\rangle \times \langle B,B\rangle$ factorizes uniquely through σ_f : $\{f,f\}_l \to \langle A,A\rangle \times \langle B,B\rangle$; that is, there is a unique $\gamma : T \to \{f,f\}_l$ with $\partial_0\gamma = \alpha$ and $\partial_1\gamma = \beta$, where α and β are the componenets of δ. Consider now a monad modification $\zeta : \delta_1 \to \delta_2$; it corresponds by Lemma 2.1 to 2-cells $\xi : a_1 \to a_2$ and $\eta : b_1 \to b_2$ for which $(1,\xi) : (A,a_2) \to (A,a_1)$ and $(1,\eta) : (B,b_2) \to (B,b_1)$ are lax T-morphisms. A monad modification $\chi : \gamma_1 \to \gamma_2$, however, corresponds by Lemma 2.2 to such a pair ξ,η satisfying the further condition $(f,\bar{f}_1)(1,\xi) = (1,\eta)(f,\bar{f}_2)$. But this condition is automatically satisfied, since T satisfies (IEL); for each of $f1$ and $1f$ is the isomorphism f. In other words, if T is such that $2\text{-Mnd}(\mathcal{K})(T,\sigma_f) : 2\text{-Mnd}(\mathcal{K})(T,\{f,f\}_l) \to 2\text{-Mnd}(\mathcal{K})(T,\langle A,A\rangle \times \langle B,B\rangle)$ is a bijection on objects for each invertible f, then in fact $2\text{-Mnd}(\mathcal{K})(T,\sigma_f)$ is an isomorphism of categories. Similarly, when it is injective on objects, as in the case of (IUL), it is in fact a monomorphism of categories; and so on.

If $2\text{-Mnd}(\mathcal{K})$ admitted the tensor product $\mathbb{2}\cdot T$, this last observation would assert that the 2-monads satisfying (IEL) are closed under the operation $\mathbb{2}\cdot(\ )$; and similarly for each of the other fourteen conditions. In the cases (XUZ) and (2Z), where the closedness under jointly-epimorphic families leads by Lemma 8.1, when $\mathcal{K}$ is complete, to coreflectiveness in $(2\text{-Mnd}(\mathcal{K}))_0$, we could then appeal to Theorem 4.85 of [16] to deduce the coreflectiveness of such T in $2\text{-Mnd}(\mathcal{K})$. This approach fails as it stands, since we know nothing about the existence in $2\text{-Mnd}(\mathcal{K})$ of $\mathbb{2}\cdot T$ for a general T; but in the following proof we overcome this difficulty by "imagining a proof using $\mathbb{2}\cdot T$" and then eliminating the occurrences of $\mathbb{2}\cdot T$ by using 2-cells $\theta : g \to h : T \to S$ in place of morphisms $f : \mathbb{2}\cdot T \to S$.

8.4. PROPOSITION. *Let $\mathcal{K}$ be complete and admit all intersections of subobjects, and let $\mathcal{B}$ be the full sub-2-category of* $2\text{-Mnd}(\mathcal{K})$ *given by those T satisfying some subset of the nine conditions given by the* (AUZ) *and the* (2Z)*. Then $\mathcal{B}$ is coreflective in* $2\text{-Mnd}(\mathcal{K})$*, with the counit a strong monomorphism.*

Proof. $\mathcal{B}$ consists either of the objects B coorthogonal to a family $(j : X_j \to Y_j)$ of strong monomorphisms, or else of the objects weakly coorthogonal to an unrestricted family $(j : X_j \to Y_j)$. In either case one easily sees — using the observation above whereby $B \in \mathcal{B}$ "satisfies the condition in the 2-categorical sense" — that, given a family $(\alpha_i : g_i \to h_i : B_i \to C)$ of 2-cells in 2-Mnd($\mathcal{K}$) with each $B_i \in \mathcal{B}$, which is "jointly epimorphic" in the sense that there is no strong subobject $z : D \to C$ of C through which each $\alpha_i : g_i \to h_i$ factorizes as $z\alpha_i' : zg_i' \to zh_i'$, we must have $C \in \mathcal{B}$. That being so, we construct the coreflection PA of A by considering the family of all 2-cells of the form $\alpha : g \to h : B \to A$ with $B \in \mathcal{B}$, and taking for $\epsilon : PA \to A$ the smallest strong subobject of A through which each $\alpha : g \to h$ factorizes. ∎

We have been unable to go further without more restrictive conditions on the 2-category $\mathcal{K}$. We shall henceforth suppose it to be *locally finitely presentable as a 2-category*, in the sense of Kelly [17]; this covers such important 2-categories as **Cat**, $\mathbf{Cat}_g$, $[\mathcal{A}, \mathbf{Cat}]$, and $[\mathcal{A}, \mathbf{Cat}_g]$, for a small category $\mathcal{A}$, as well as the 2-category Lex$[\mathcal{T}, \mathbf{Cat}]$ of models for a "2-limit theory" $\mathcal{T}$. Even for such a $\mathcal{K}$, we cannot establish the coreflectiveness in 2-Mnd($\mathcal{K}$) of the property-like 2-monads, or of the fully property-like ones, or of the lax-idempotent ones: one problem is that little is known of colimits $colim_i T_i$ (see Section 29 of [15]) in 2-Mnd($\mathcal{K}$), except in the case where each T_i *has some rank*. We do have positive results, however, when we replace 2-Mnd($\mathcal{K}$) by its full sub-2-category 2-Mnd$_\alpha(\mathcal{K})$ given by the 2-monads of rank α — the most important case of which is the full sub-2-category 2-Mnd$_{fin}(\mathcal{K})$ given by the *finitary* 2-monads. Recall here that, for a regular cardinal α, an endo-2-functor T (or a 2-monad (T, m, i)) is said *to have rank* α (or, in a more modern terminology, *to be* α*-accessible*) when T preserves α-filtered colimits; and to be *finitary* when it has rank ω, the first infinite cardinal — that is, when it preserves filtered colimits. In fact we shall set out our arguments below just for this important case of finitary 2-monads, leaving to the reader the straightforward extension to 2-monads of some given rank α.

Recall from [17] that an object c of $\mathcal{K}$ is said to be *finitely presentable* when $\mathcal{K}(c, -) : \mathcal{K} \to \mathbf{Cat}$ is finitary. The full sub-2-category $\mathcal{K}_f$ of $\mathcal{K}$ given by the finitely-presentable objects is small and finitely cocomplete, and we recover $\mathcal{K}$ as Lex$[\mathcal{K}_f^{op}, \mathbf{Cat}]$, the full sub-2-category of $[\mathcal{K}_f^{op}, \mathbf{Cat}]$ given by those presheaves that preserve "finite 2-categorical limits" in the sense of [17, Section 4]. For such a 2-category $\mathcal{K}$, it was shown in [17] that the underlying ordinary category $\mathcal{K}_0$ is locally finitely presentable in the classical sense, with $(\mathcal{K}_0)_f = (\mathcal{K}_f)_0$, and with equivalences $\mathcal{K}_0 \simeq [\mathcal{K}_{0f}^{op}, \mathbf{Set}] \simeq [\mathcal{K}_f^{op}, \mathbf{Cat}]_0$.

Central to our considerations below are the following results of [17, Prop. 7.6]. Write Fin$[\mathcal{K}, \mathcal{K}]$ for the full sub-2-category of $[\mathcal{K}, \mathcal{K}]$ given by the finitary endo-2-functors. An endo-2-functor T of $\mathcal{K}$ is finitary precisely when it is the left Kan extension (in the 2-categorical sense) of its restriction to $\mathcal{K}_f$; indeed restriction along the inclusion $J : \mathcal{K}_f \to \mathcal{K}$ induces an equivalence of 2-categories Fin$[\mathcal{K}, \mathcal{K}] \to [\mathcal{K}_f, \mathcal{K}]$, an inverse of which sends $P : \mathcal{K}_f \to \mathcal{K}$ to its left Kan extension Lan$_J P$. Since $\mathcal{K}_f$ is small and $\mathcal{K}$ is locally finitely presentable, the 2-category $[\mathcal{K}_f, \mathcal{K}]$ is itself locally finitely presentable by [17, Example 3.4]; accordingly the equivalent 2-category Fin$[\mathcal{K}, \mathcal{K}]$ is locally finitely presentable (and thus much better behaved than $[\mathcal{K}, \mathcal{K}]$, which is not even locally small).

Since the finitary endo-2-functors of $\mathcal{K}$ are closed under composition, the 2-category Fin$[\mathcal{K}, \mathcal{K}]$ has a monoidal structure with the composition as its tensor product. Under the equivalence above, this induces a monoidal structure $([\mathcal{K}_f, \mathcal{K}], \circ, J)$ on $[\mathcal{K}_f, \mathcal{K}]$. Writing $\mathcal{M}$ for the 2-category 2-Mnd$_{fin}(\mathcal{K})$ = MonFin$[\mathcal{K}, \mathcal{K}]$ of finitary 2-monads on $\mathcal{K}$, we have an equivalence $\mathcal{M} \simeq \mathrm{Mon}[\mathcal{K}_f, \mathcal{K}]$ of monoidal 2-categories. The first step towards our further results is given by:

8.5. Proposition. *When $\mathcal{K}$ is locally finitely presentable, the full sub-2-category $\mathcal{M}$ = 2-Mnd$_{fin}(\mathcal{K})$ of 2-Mnd$(\mathcal{K})$ consisting of the finitary 2-monads is coreflective in 2-Mnd$(\mathcal{K})$. Moreover $\mathcal{M}$ is itself a locally finitely presentable 2-category.*

Proof. The inclusion $\mathrm{Lan}_J : [\mathcal{K}_f, \mathcal{K}] \to [\mathcal{K}, \mathcal{K}]$ preserves to within isomorphism the "tensor products", by the very definition of the tensor product $\circ$ on $[\mathcal{K}_f, \mathcal{K}]$. It follows easily that the right adjoint $(\ )J : [\mathcal{K}, \mathcal{K}] \to [\mathcal{K}_f, \mathcal{K}]$ has the structure of a monoidal functor, with a canonical comparison $TJ \circ SJ \to TSJ$. In these circumstances the adjunction $\mathrm{Lan}_J \dashv (\ )J : [\mathcal{K}, \mathcal{K}] \to [\mathcal{K}_f, \mathcal{K}]$ passes to the monoids to give an adjunction $L \dashv R$: $\mathrm{Mon}[\mathcal{K}, \mathcal{K}] \to \mathrm{Mon}[\mathcal{K}_f, \mathcal{K}]$; and the L here, modulo the equivalence $\mathcal{M} \simeq \mathrm{Mon}[\mathcal{K}_f, \mathcal{K}]$, is the inclusion of $\mathcal{M}$ in 2-Mnd$(\mathcal{K})$; so that $\mathcal{M}$ is indeed coreflective. Now consider the forgetful 2-functor $V : \mathrm{Mon}[\mathcal{K}_f, \mathcal{K}] \to [\mathcal{K}_f, \mathcal{K}]$. This is easily seen to create all limits (including weighted ones), and so in particular to create the cotensor products $T^{\mathbf{2}}$. By [16, Thm 4.85], therefore, V will have a left adjoint if its underlying ordinary functor V_0 does so — that is, if free monoids exist in the ordinary sense; and in fact they do exist, by [15, Thm 23.3]. It then follows easily that V is monadic, using the Beck criterion (which works perfectly well in the context of enriched categories); the details of this argument for the monadicity of V (actually of V_0) are given in [20, Section 4]. In that same section of [20] it is shown that V is finitary; thus the 2-monad $M = VG$, where G is the left adjoint of V, is finitary. We conclude that $\mathcal{M} \simeq \mathrm{Mon}[\mathcal{K}_f, \mathcal{K}]$ is further equivalent to the Eilenberg-Moore 2-category $[\mathcal{K}_f, \mathcal{K}]^M$ for a finitary 2-monad M. However $\mathcal{A}^M$ is locally finitely presentable when $\mathcal{A}$ is so if the 2-monad M has some rank: for the argument in the ordinary-category case, given in [13, Satz 10.3], applies unchanged in the enriched context. ∎

8.6. Remark. We shall need the following observation: *given locally finitely presentable categories $\mathcal{M}$ and $\mathcal{N}$, along with left-adjoint functors $P, Q : \mathcal{M} \to \mathcal{N}$ and a 2-cell $\alpha : P \to Q$, consider the inverter in* ***CAT*** *of α, namely the full subcategory $\mathcal{P}$ of $\mathcal{M}$ given by those objects $M \in \mathcal{M}$ for which α_M is invertible. Then $\mathcal{P}$ is coreflective in $\mathcal{M}$, and is itself locally presentable (although not necessarily locally finitely presentable).* Although first proved as Proposition 3.14 of Bird's thesis [3], which remains unpublished, this result can also be seen as a special case of [24, Thm 5.16], or of [1, Thm 2.77].

8.7. Theorem. *Let $\mathcal{K}$ be a locally finitely presentable 2-category, and let $\mathcal{M}$ again denote the sub-2-category 2-Mnd$_{fin}(\mathcal{K})$ of 2-Mnd$(\mathcal{K})$ given by the finitary 2-monads on $\mathcal{K}$. Define full sub-2-categories $\mathcal{L} \subset \mathcal{F} \subset \mathcal{P} \subset \mathcal{M}$ as follows: $\mathcal{P}$ consists of the finitary 2-monads on $\mathcal{K}$ that are property-like, $\mathcal{F}$ consists of those that are fully property-like, and $\mathcal{L}$ of those that are lax-idempotent. Then each of $\mathcal{L}$, $\mathcal{F}$, and $\mathcal{P}$ is coreflective in $\mathcal{M}$ and hence (by Proposition 8.5) coreflective in 2-Mnd$(\mathcal{K})$; and moreover $\mathcal{P}$ is itself locally presentable.*

Proof. We first consider $\mathcal{P}$. By Theorem 4.2, a 2-monad T in $\mathcal{M}$ lies in $\mathcal{P}$ if and only if, for any monad morphisms $f, g : T \to S$, there is a unique monad modification $\chi : f \to g$. Write $j_S : S' \to S$ for the coreflection of 2-Mnd($\mathcal{K}$) in $\mathcal{M}$ given by Proposition 8.5. Then to give $f, g : T \to S$ is just to give $f', g' : T \to S'$ with $f = jf'$ and $g = jg'$; while to give $\chi : f \to g$ is just to give $\chi' : f' \to g'$ with $\chi = j\chi'$. It follows that $T \in \mathcal{M}$ lies in $\mathcal{P}$ if and only if, *for each* $S \in \mathcal{M}$, we have the property that each pair $f, g : T \to S$ in $\mathcal{M}$ admits a unique $\chi : f \to g$ in $\mathcal{M}$. Since $\mathcal{M}$ is locally finitely presentable by Proposition 8.5 and hence cocomplete as a 2-category, it admits the tensor products $2 \cdot T$ and $\mathbb{2} \cdot T$; so that the truth of the condition for each $S \in \mathcal{M}$ becomes, as in (xvi) of Theorem 4.2, the invertibility of $\kappa \cdot T : 2 \cdot T \to \mathbb{2} \cdot T$. That is to say, $\mathcal{P}$ is the inverter of $\kappa \cdot (\ ) : 2 \cdot (\ ) \to \mathbb{2} \cdot (\ )$; and now Remark 8.6 gives the coreflectiveness of $\mathcal{P}_0$ in $\mathcal{M}_0$. The assertion about 2-cells that would enrich this to the coreflectiveness of $\mathcal{P}$ in $\mathcal{M}$ is in fact trivial, precisely because each $\mathcal{P}(T, S)(f, g)$ with $T \in \mathcal{P}$ is a singleton; so it is indeed the case that $\mathcal{P}$ is coreflective in $\mathcal{M}$ and hence in 2-Mnd($\mathcal{K}$).

When we turn to $\mathcal{F}$ and to $\mathcal{L}$, it is again the case — since these are contained in $\mathcal{P}$— that the 2-dimensional aspect of the coreflection is trivial; so that it suffices to establish the coreflectiveness in $\mathcal{P}_0$ of $\mathcal{F}_0$ and of $\mathcal{L}_0$. Now $\mathcal{F}_0$ consists of those objects T of $\mathcal{P}_0$ that satisfy (2L) and (2C): which by Remark 5.5 are those T in $\mathcal{P}_0$ that are coorthogonal in (2-Mnd($\mathcal{K}$))$_0$ to the strong monomorphisms ν_ρ (for (2L)) along with further strong monomorphisms $\bar{\nu}_\rho$ (for (2C)). Equally, therefore, $\mathcal{F}_0$ consists of those T in $\mathcal{P}_0$ that are coorthogonal in $\mathcal{P}_0$ to the coreflections in $\mathcal{P}_0$ of the ν_ρ and the $\bar{\nu}_\rho$; but these coreflections — let us call them ν^*_ρ and $\bar{\nu}^*_\rho$ — are strong monomorphisms in $\mathcal{P}_0$, since right adjoints preserve strong monomorphisms. Since the locally presentable category $\mathcal{P}_0$ is complete and well-powered, it certainly admits all intersections of subobjects; so that we may use Lemma 8.1 to conclude that $\mathcal{F}_0$ is coreflective in $\mathcal{P}_0$.

It remains to show that $\mathcal{L}_0$ is coreflective in $\mathcal{P}_0$. Since the lax-idempotent 2-monads are those satisfying (AEL), $\mathcal{L}_0$ consists by our remarks above of those T in $\mathcal{P}_0$ that are coorthogonal in (2-Mnd($\mathcal{K}$))$_0$ to each $\sigma : \{f, f\}_l \to \langle A, A\rangle \times \langle B, B\rangle$. So $\mathcal{L}_0$ is equally the full subcategory of $\mathcal{P}_0$ given by the objects coorthogonal to the σ^*_f, where σ^*_f is the coreflection into $\mathcal{P}_0$ of the arrow σ_f of 2-Mnd($\mathcal{K}$). The desired coreflectiveness of $\mathcal{L}_0$ in $\mathcal{P}_0$ now follows by Lemma 8.1, if we show that each σ^*_f is monomorphic. Equivalently, we must show for a property-like T that every monad morphism $\delta : T \to \langle A, A\rangle \times \langle B, B\rangle$ (say with components α and β) is of the form $\sigma_f\gamma$ for *at most one* monad morphism $\gamma : T \to \{f, f\}_l$. This, in turn, is the statement that, for given T-actions $a : TA \to A$ and $b : TB \to B$, and a given $f : A \to B$, there is at most one 2-cell $\bar{f}$ for which $(f, \bar{f})$ is a lax T-morphism $(A, a) \to (B, b)$; or equivalently the statement (AUL), which is true of a property-like T by Theorem 4.2. This finally completes the proof. ∎

8.8. Remark. As we observed earlier there is, for each regular cardinal α, an analogue of Theorem 8.7 with 2-Mnd$_{fin}$($\mathcal{K}$) replaced by the 2-category 2-Mnd$_\alpha$($\mathcal{K}$) whose objects are the endo-2-functors of rank α. This raises various questions to which we have so far found no answer, such as: does the coreflection of 2-Mnd$_\alpha$($\mathcal{K}$) into the property-likes carry finitary 2-monads into finitary ones? Perhaps the simplest questions of this kind concern the coreflections of Theorem 8.2: does the coreflection of an ordinary monad into the property-likes, or into the idempotents, preserve finitariness?

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