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POINTS OF AFFINE CATEGORIES AND ADDITIVITY

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Abstract. A category $\mathbf{C}$ is additive if and only if, for every object B of $\mathbf{C}$, the category $\mathrm{Pt}(\mathbf{C}, B)$ of pointed objects in the comma category $(\mathbf{C}, B)$ is canonically equivalent to $\mathbf{C}$. We reformulate the proof of this known result in order to obtain a stronger one that uses not all objects of B of $\mathbf{C}$, but only a conveniently defined generating class $\mathbf{S}$. If $\mathbf{C}$ is a variety of universal algebras, then one can take $\mathbf{S}$ to be the class consisting of any single free algebra on a non-empty set.

1. Additive Objects

For any category $\mathbf{X}$ with a terminal object $\mathbf{1}$, we denote with

$$\mathrm{Pt}(\mathbf{X})$$

the category of (global) points $x: \mathbf{1} \longrightarrow X$ of $\mathbf{X}$, with maps the obvious commutative triangles. Observe that the category $\mathrm{Pt}(\mathbf{X})$ is pointed, i.e. the terminal is also initial, and recall that for any category $\mathbf{C}$ and any object B of $\mathbf{C}$, the the category of objects over B, i.e. the comma cateory $(\mathbf{C}, B)$ has a terminal object, namely the identity map of B. Hence the objects of the pointed category $\mathrm{Pt}(\mathbf{C}, B)$ are triples (A, α, β), where $\alpha: A \longrightarrow B$ and $\beta: B \longrightarrow A$ are maps in $\mathbf{C}$ such that $\alpha\beta = 1$ and the maps are the maps in $\mathbf{C}$ for wich the two resulting triangles commute. We will refer to α as the "structural map".

We now revisit a result due to D. Bourn (see [Bourn 1991]), which inverts a remark contained in [Carboni 1989], according to which, for any additive category $\mathbf{C}$ with kernels, and any object B of $\mathbf{C}$ the functor

$$(\ \ ) \oplus B: \mathbf{C} \longrightarrow \mathrm{Pt}(\mathbf{C}, B)$$

is in fact an equivalence, the adjoint inverse being the kernel of the structural map to B. We give a proof of Bourn's result, according to which if for a (necessarily) pointed category $\mathbf{C}$ with finite limits and finite sums, all the functors $(\ \ ) + B$ are equivalences, then $\mathbf{C}$ is additive. The reason is not just for the self-completeness of this note, but also because we will give a proof more appropriate for the discussion in the next section.

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Let $\mathbf{C}$ be a pointed category with finite limits and finite coproducts. For an object B in $\mathbf{C}$, consider the adjunction

$$(F, G, \eta, \epsilon) \colon \mathbf{C} \longrightarrow \mathrm{Pt}(\mathbf{C}, B)$$

in which:

1. the functor $F \colon \mathbf{C} \longrightarrow \mathrm{Pt}(\mathbf{C}, B)$ is defined by

$$F(X) = (B + X, [1, 0], i_1)\,,$$

where $[1, 0]$ is the morphism $B + X \longrightarrow B$ induced by the identity $1 \colon B \longrightarrow B$ and the zero map $0 \colon X \longrightarrow B$, and i_1 is the injection $i_1 \colon B \longrightarrow B + X$;

2. the functor $G \colon \mathrm{Pt}(\mathbf{C}, B) \longrightarrow \mathbf{C}$ is given by the kernel of the structural map $A \longrightarrow B$;

3. $\eta_X \colon X \longrightarrow GF(X) = B \flat X = Ker([1, 0] \colon B + X \longrightarrow B)$ is the unique morphism $X \longrightarrow B \flat X$ for which the composite

$$X \longrightarrow B \flat X \longrightarrow B + X$$

is the second injection.

4. $\epsilon_{(A,\alpha,\beta)} = [\beta, ker(\alpha)] \colon FG(A, \alpha, \beta) = (B + Ker(\alpha), [1, 0], i_1) \longrightarrow (A, \alpha, \beta)$.

1.1. Proposition. *If the adjunction (F, G, η, ϵ) satisfies one of the following equivalent conditions*

a) *is an equivalence,*

b) *G is full and faithful,*

c) *ϵ is an isomorphism,*

then the canonical morphism

$$B + X \longrightarrow B \times X$$

is an isomorphism for every object X in $\mathbf{C}$.

Proof. Take $(A, \alpha, \beta) = (B \times X, \pi_1, \langle 1, 0\rangle)$, where $\pi_1 \colon B \times X \longrightarrow B$ is the projection and $\langle 1, 0\rangle$ is the pair given by the identity and the zero maps. Since the kernel of $\pi_1 \colon B \times X \longrightarrow B$ can be identified with $\langle 0, 1\rangle \colon X \longrightarrow B + X$, it is easy to see that $\epsilon_{(A,\alpha,\beta)}$ can be identified with the morphism $B + X \longrightarrow B \times X$, which therefore is an isomorphism as desired.

As for the equivalence with the other two conditions, just observe that from the previous proof it follows that, instead of requiring the adiunction be an equivalence, we could have required just ϵ be an isomorphism. Moreover, observe that we always have $G(B \times X, \pi_1, \langle 1, 0\rangle) \simeq X$, so that G is always essentially surjective on objects. ∎

From now on, **B** will denote the full subcategory of **C** with objects all B in **C**, for which the morphism $B + X \longrightarrow B \times X$ is an isomorphism for every X in **C**.

1.2. PROPOSITION. **B** *is closed (in* **C***) under finite products and finite coproducts, and has a unique enrichment in the category of commutative monoids.*

PROOF. We observe:

a) Obviously, 0 is in **B**.

b) Given A and B in **B** and X in **C**, we have canonical isomorphisms

$$(A + B) + X \simeq A + (B + X) \simeq A \times (B \times X) \simeq (A \times B) \times X \simeq (A + B) \times X,$$

which prove that **B** is closed under finite sums.

c)) If A and B in **B**, then their product in **C** is isomorphic to their coproduct. Therefore b) implies that **B** is closed under finite products.

d) As follows from c) and the definition of **B**, for every two objects A and B in **B**, the canonical morphism $A + B \longrightarrow A \times B$ inside **B** (and in the sense of **B**) is an isomorphism. Therefore, **B** has a unique enrichment in the category of commutative monoids, as desired. ∎

From now on, **A** will denote the full subcategory of **C** with objects all B in **C**, for which the adjunction (F, G, η, ϵ) is an equivalence. As follows from Proposition 1, **A** is contained in **B**. Note also, that since **B** is a full subcategory of **C** closed under finite products and enriched in the category of commutative monoids, every object of **B** has a unique (commutative) internal monoid structure in **C**; in particular the same is true for every object of **A**.

1.3. PROPOSITION. *Every object of* **A** *has a unique internal abelian group structure in* **C**.

PROOF. For an object B in **A**, take $(A, \alpha, \beta) = (B \times B, \pi_1, \langle 1, 1\rangle)$ in the notation above. Since ϵ is an isomorphism, so is the morphism

$\epsilon_{(A,\alpha,\beta)} = [\langle 1, 1\rangle, \langle 0, 1\rangle] \colon FG(A, \alpha, \beta) =$
$= (B + B, [1, 0], i_1) \longrightarrow (B \times B, \pi_1, \langle 1, 1\rangle).$

However this is now happening inside of a category in which the finite coproducts are canonically isomorphic to finite products (namely inside **B**); therefore the above morphism is the same as

$$\langle 1, 1\rangle + \langle 0, 1\rangle \colon B \times B \longrightarrow B \times B\,,$$

and, of course, this morphism is an isomorphism if and only if B is a group. ∎

This result of course implies the quoted result of Bourn, but it is phrased in such a way to allow us to apply it to the next question.

2. Involving Generators

For a ring R, the category $\mathrm{Aff}(R)$ of affine spaces over R can be described as the category of all pairs (A, a), where A is an R-module and a an element of A. That is, one can *define* $\mathrm{Aff}(R)$ as the comma category $(R, R\text{-Mod})$. Furthermore, we can recover the category R-Mod from $\mathrm{Aff}(R)$ as the comma category $(\mathrm{Aff}(R), R) = ((R, R\text{-Mod}), R)$. Or, we could say that R-Mod can be recovered from the comma category $(R\text{-Mod}, R)$ as the category $(R, (R\text{-Mod}, R)) = \mathrm{Pt}(R\text{-Mod}, R)$ of its points. Does this tell us something about the additivity of R-Mod? That is, given a variety $\mathbf{V}$ of universal algebras, is it true that if $\mathbf{V}$ is canonically equivalent to $\mathrm{Pt}(\mathbf{V})$, where R is the free algebra on one element, then $\mathbf{V}$ is additive? And if so, is this an instance of a general-categorical fact? In other words, we wish to find out the minimal condition on a class $\mathbf{S}$ of objects of a pointed $\mathbf{C}$, such that when the canonical functors from $\mathbf{C}$ the the categories of points of the slices over the objects of $\mathbf{S}$ are equivalences, then $\mathbf{C}$ is additive. Clearly we need an appropriate notion of a class $\mathbf{S}$ of generators for such categories $\mathbf{C}$, and we propose the following:

2.1. Definition. *A class* $\mathbf{S}$ *of objects in* $\mathbf{C}$ *is said to be a* class of generators *if it satisfies the following condition: Let* $\mathbf{X}$ *be generated by* $\mathbf{S}$ *as a full subcategory in* $\mathbf{C}$ *closed under finite products and finite coproducts. Then the restriction of the Yoneda functor*

$$\mathbf{C} \longrightarrow \mathit{Sets}^{\mathbf{X}^{op}}$$

sending any object C *to the hom functor*

$$\hom_{\mathbf{C}}(-, C) \colon \mathbf{X}^{op} \longrightarrow \mathit{Sets}$$

is full and faithful.

2.2. Theorem. *If* $\mathbf{C}$ *admits a class of generators contained in* $\mathbf{A}$*, then* $\mathbf{C}$ *is additive.*

Proof. Take $\mathbf{S}$ and $\mathbf{X}$ as in the definition above, assuming that $\mathbf{S}$ is contained in $\mathbf{A}$, and observe:

a) Since $\mathbf{S}$ is contained in $\mathbf{B}$ and $\mathbf{B}$ is closed under under finite products and finite coproducts in $\mathbf{C}$, then $\mathbf{X}$ is contained in $\mathbf{B}$.

b) Since every object in $\mathbf{S}$ has a group structure, a) tells that $\mathbf{X}$ is additive.

c) Since the restriction of the Yoneda functor is full and faithful, it is easy to see that b) implies the additivity of $\mathbf{C}$ (just consider the intermediate category of product preserving functors $\mathbf{X}^{op} \longrightarrow$ Sets). ∎

Note that if $\mathbf{C}$ were a variety of universal algebras, and $\mathbf{S}$ is the free algebras on one generator, then the restriction of the Yoneda functor $\mathbf{C} \longrightarrow \mathrm{Sets}^{\mathbf{X}^{op}}$ is full and faithful, but $\mathbf{C} \longrightarrow \mathrm{Sets}^{\mathbf{S}^{op}}$ is not. Note also that although Definition 4 uses both (finite) products and coproducts, to obtain $\mathbf{X}$ from $\mathbf{S}$, under the assumptions of Theorem 5 these products and coproducts coincide. The only reason of involving both products and coproducts is that in general neither of these two concepts is superior to the other.

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