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DIRECTED HOMOTOPY THEORY, II. HOMOTOPY CONSTRUCTS

MARCO GRANDIS

ABSTRACT.

Directed Algebraic Topology studies phenomena where privileged directions appear, derived from the analysis of concurrency, traffic networks, space-time models, etc.

This is the sequel of a paper, 'Directed homotopy theory, I. The fundamental category', where we introduced *directed spaces*, their non reversible homotopies and their fundamental category. Here we study some basic constructs of homotopy, like homotopy pushouts and pullbacks, mapping cones and homotopy fibres, suspensions and loops, cofibre and fibre sequences.

Introduction

Directed Algebraic Topology is a recent subject, arising from domains where privileged directions appear, like concurrent processes, traffic networks, space-time models, etc. (cf. [2, 3, 4, 5, 9, 10]). Its domain should be distinguished from classical Algebraic Topology by the principle that *directed spaces have privileged directions, and directed paths therein need not be reversible.* Its homotopical tools will also be 'non-reversible': *directed homotopies* and *fundamental category* instead of ordinary homotopies and fundamental groupoid.

This is a sequel of a paper which will be cited as Part I [11]; I.1 (resp. I.1.2, or I.1.2.3) will refer to Section 1 of Part I (resp. its Subsection 1.2, or item (3) in the latter).

In Part I, we introduced *directed spaces*, their directed homotopies and their fundamental category, including a 'Seifert-van Kampen' type theorem, to compute it. The notion of 'directed space' which we are using is a topological space X equipped with a family dX of 'directed paths' $[0, 1] \to X$, containing all constant paths and closed under increasing reparametrisation and concatenation. Such objects, called *directed spaces* or *d-spaces*, with the obvious *d-maps* - preserving the assigned paths - form a category d**Top** which has general properties similar to **Top**. (The prefixes d, ↑ are used to distinguish a directed notion from the corresponding 'reversible' one.) Relations of d-spaces with preordered spaces, locally preordered spaces, bitopological spaces and generalised metric spaces with 'asymmetric' distance have been discussed in I.1 and I.4.

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Directed homotopies are based on the standard directed interval $\uparrow\mathbf{I}$, the *directed cylinder functor*, $\uparrow I(X) = X\times\uparrow\mathbf{I}$, and its right adjoint, the *directed path functor*, $\uparrow P(X) = X^{\uparrow\mathbf{I}}$. Such functors, with a structure consisting of faces, degeneracy, connections and interchange, satisfy the axioms of an IP-*homotopical category*, as studied in [7] for a different case of directed homotopy, cochain algebras; moreover, for d-spaces, paths and homotopies can be concatenated.

We develop here a study of homotopy pushouts and pullbacks (Section 1); the main results deal with their 2-dimensional universal property and its consequences, and are similar to certain general results of [G1] (for categories equipped with 'formal homotopies'), yet more delicate because here we have to take care of the direction of homotopies. In Sections 2-3, mapping cones and suspensions are dealt with, as well as homotopy fibres and loop-objects (for *pointed* d-spaces, of course); higher homotopy monoids $\uparrow\pi_n(X, x)$ are introduced in 3.4. Combining the present results with the general theory developed in [7], we obtain the cofibre sequence of a map and the fibre sequence of a pointed map, including their 'exactness property' (Theorem 2.5), by comparison with sequences of iterated mapping (co)cones, alternatively *lower* or *upper*. Note that, even if paths *in* a d-space X cannot be *reversed*, generally, they can nevertheless be *reflected* in the *opposite* object $RX = X^{\mathrm{op}}$; thus, lower and upper cones determine each other (2.1). The fibre sequence of a pointed map (3.3) produces a sequence of higher homotopy monoids which is *not* exact, generally (3.5); but it has already been observed in Part I that the homotopy monoids $\uparrow\pi_1(X, x)$ contain only a fragment of the fundamental category $\uparrow\Pi_1(X)$: higher dimensional properties should probably be studied by higher fundamental *categories* (as introduced in [9] for simplicial sets). Finally, in Section 4, we shall see how, on 'comma categories' d$\mathbf{Top}\backslash A$ of d-spaces under a discrete object, the d-homotopy invariance of the fundamental functor $\uparrow\Pi_1$ is strict 'on the base points'.

As in Part I, category theory intervenes at an elementary level. Some basic facts are of frequent use: all (categorical) *limits* (generalising cartesian products and projective limits) can be constructed from products and equalisers; dually, all *colimits* (generalising sums and injective limits) can be constructed from sums and coequalisers. Left adjoint functors preserve all the existing colimits, while right adjoints preserve limits; $F \dashv G$ means that F is left adjoint to G. Comma categories are only used in the last section (see [12, 1]). A *map* between topological spaces is a continuous mapping. A homotopy φ between maps $f, g\colon X \to Y$ can be written as $\varphi\colon f \to g\colon X \to Y$, or $\varphi\colon f \to g$, or $\varphi\colon X \Rightarrow Y$. An *order* relation is reflexive, transitive and anti-symmetric; a mapping which preserves such relations is said to be *increasing* (always used in the weak sense). The index α takes values 0, 1, written $-, +$ in superscripts.

1. Directed homotopy pushouts and pullbacks

Homotopy pushouts of d-spaces can be constructed in the usual way, from the directed cylinder. We shall always work with the standard ones, determined *up to isomorphism* by strict universal properties. The main results, here, concern their 2-dimensional properties.

1.1. A REVIEW OF DIRECTED SPACES. A *directed topological space* $X = (X, dX)$, or d-space (I.1.1), is a topological space equipped with a set dX of (continuous) maps $a: \mathbf{I} \to X$, defined on the standard interval $\mathbf{I} = [0,1]$; these maps, called *directed paths* or *d-paths*, must contain all constant paths and be closed under (weakly) increasing reparametrisation and concatenation. The d-space X is thus equipped with a *path-preorder* $x \preceq x'$, defined by the existence of a directed path from x to x'.

A *directed map* $f: X \to Y$ (or *d-map*, or *map* of d-spaces) is a continuous mapping between d-spaces which preserves the directed paths: if $a \in dX$, then $fa \in dY$.

The category of d-spaces is written as d**Top**. It has all limits and colimits, constructed as in **Top** and equipped with the initial or final d-structure for the structural maps; for instance a path $\mathbf{I} \to \Pi X_i$ is directed if and only if all its components $\mathbf{I} \to X_i$ are so. The forgetful functor U: d**Top** $\to$ **Top** preserves thus all limits and colimits; a topological space is generally viewed as a d-space by its *natural* structure, where all paths are directed (via the right adjoint to U, I.1.1).

Reversing d-paths, by the involution $r(t) = 1 - t$, yields the *reflected*, or *opposite*, d-space $RX = X^{\mathrm{op}}$, where $a \in d(X^{\mathrm{op}})$ if and only if $a^{\mathrm{op}} = ar \in dX$. A d-space is *symmetric* if it is invariant under reflection. More generally, it is *reflexive*, or *self-dual*, if it is isomorphic to its reflection.

The *directed real line*, or *d-line* $\uparrow\mathbf{R}$, is the Euclidean line with directed paths given by the (weakly) increasing maps $\mathbf{I} \to \mathbf{R}$. Its cartesian power in d**Top**, the *n-dimensional real d-space* $\uparrow\mathbf{R}^n$ is similarly described (with respect to the product order, $x \leq y$ if $x_i \leq y_i$ for all i). The *standard d-interval* $\uparrow\mathbf{I} = \uparrow[0,1]$ has the subspace structure of the d-line; the *standard d-cube* $\uparrow\mathbf{I}^n$ is its n-th power, and a subspace of $\uparrow\mathbf{R}^n$. These d-spaces are not symmetric (for $n > 0$), yet reflexive. The *standard directed circle* $\uparrow\mathbf{S}^1 = \uparrow\mathbf{I}/\partial\mathbf{I}$ has the obvious 'counter-clockwise' d-structure; but we also consider the natural circle $\mathbf{S}^1$ and the *ordered circle* $\uparrow\mathbf{O}^1 \subset \mathbf{R} \times \uparrow\mathbf{R}$ (I.1.2); for higher spheres, see 2.3, 3.2.

The directed interval $\uparrow\mathbf{I} = \uparrow[0,1]$ is a lattice in d**Top**; its structure (I.2.1) consists of two faces (∂^-, ∂^+), a degeneracy (e), two connections or main operations (g^-, g^+) and an interchange (s)

$$\{*\} \underset{e}{\overset{\partial^\alpha}{\rightleftarrows}} \uparrow\mathbf{I} \overset{g^\alpha}{\longleftarrow} \uparrow\mathbf{I}^2 \qquad\qquad \uparrow\mathbf{I}^2 \xrightarrow{\ s\ } \uparrow\mathbf{I}^2 \tag{1}$$

$$\begin{array}{lll} \partial^-(*) = 0, & \partial^+(*) = 1, & \\ g^-(t,t') = \max(t,t'), & g^+(t,t') = \min(t,t'), & s(t,t') = (t',t). \end{array}$$

As a consequence, the (directed) *cylinder* endofunctor of d-spaces, $\uparrow I(X) = X \times \uparrow\mathbf{I}$, has natural transformations, which are denoted by the same symbols and names

$$1 \underset{e}{\overset{\partial^\alpha}{\rightleftarrows}} \uparrow I \overset{g^\alpha}{\longleftarrow} \uparrow I^2 \qquad\qquad \uparrow I^2 \xrightarrow{\ s\ } \uparrow I^2 \tag{2}$$

and satisfy the axioms of a *cubical monad with interchange* ([8], Section 2).

The directed interval $\uparrow\mathbf{I}$ is exponentiable (Theorem I.1.7): this means that the cylinder functor $\uparrow I = -\times\uparrow\mathbf{I}$ has a right adjoint, the (directed) *path functor*, or *cocylinder* $\uparrow P$

$$\uparrow P\colon \mathrm{d}\mathbf{Top} \to \mathrm{d}\mathbf{Top}, \qquad \uparrow P(Y) = Y^{\uparrow\mathbf{I}}, \tag{3}$$

where the d-space $Y^{\uparrow\mathbf{I}}$ is the set of d-paths $\mathrm{d}\mathbf{Top}(\uparrow\mathbf{I}, Y)$ with the usual compact-open topology and the d-structure where a map $c\colon \uparrow\mathbf{I} \to \mathrm{d}\mathbf{Top}(\uparrow\mathbf{I}, Y)$ is directed if and only if, for all increasing maps $h, k\colon \uparrow\mathbf{I} \to \uparrow\mathbf{I}$, the derived path $t \mapsto c(h(t))(k(t))$ is in dY (I.2.2).

The lattice structure of $\uparrow\mathbf{I}$ in $\mathrm{d}\mathbf{Top}$ produces - contravariantly - a dual structure on $\uparrow P$ (a cubical *co*monad with interchange [8]); the derived natural transformations (*faces*, etc.) will be named and written as above, but proceed in the opposite direction and satisfy dual axioms (note that $\uparrow P^2(Y) = (Y^{\uparrow\mathbf{I}})^{\uparrow\mathbf{I}} = Y^{\uparrow\mathbf{I}^2}$, by composing adjunctions)

$$1 \underset{e}{\overset{\partial^\alpha}{\leftleftarrows\!\!\to}} \uparrow P \overset{g^\alpha}{\rightrightarrows} \uparrow P^2 \qquad\qquad \uparrow P^2 \overset{s}{\longrightarrow} \uparrow P^2 \tag{4}$$

$\partial^-(a) = a(0), \qquad \partial^+(a) = a(1), \qquad e(x)(t) = x,$
$g^-(a)(t, t') = a(\max(t, t')), \ldots$

Now, a (directed) *homotopy* $\varphi\colon f \to g\colon X \to Y$ is defined as a d-map $\varphi\colon \uparrow IX = X\times\uparrow\mathbf{I} \to Y$ whose two faces $\partial^\pm(\varphi) = \varphi.\partial^\pm\colon X \to Y$ are f and g, respectively. Equivalently, it is a d-map $X \to \uparrow P(Y) = Y^{\uparrow\mathbf{I}}$, with faces as above. A (directed) path $a\colon \uparrow\mathbf{I} \to X$ is the same as a d-path $a \in dX$, and amounts to a homotopy between two points, $a\colon x \to x'\colon \{*\} \to X$. The structure of d-homotopies (I.2.3) essentially consists of the following operations (for $u\colon X' \to X, v\colon Y \to Y', \psi\colon g \to h$)

(a) *whisker composition of maps and homotopies*:
$v{\circ}\varphi{\circ}u\colon vfu \to vgu \qquad (v{\circ}\varphi{\circ}u = v.\varphi.\uparrow Iu\colon \uparrow IX' \to Y'),$

(b) *trivial homotopies*:
$0_f\colon f \to f \qquad (0_f = fe\colon \uparrow IX \to Y),$

(c) *concatenation of homotopies*:
$\varphi + \psi\colon f \to h$ (defined via the concatenation of d-paths).

(The whisker composition will also be written by juxtaposition, when this is not ambiguous.) The category of d-spaces is an IP-*homotopical category* ([7], 2.7); loosely speaking, it has:

- adjoint endofunctors $\uparrow I \dashv \uparrow P$, with the required structure (faces, etc., satisfying the axioms);

- all pushouts (preserved by the cylinder) and all pullbacks (preserved by the cocylinder);

- initial and terminal object.

Therefore, all results of [7] for such a structure apply (as for cochain algebras). Moreover, *here we can concatenate paths and homotopies*.

Let us also briefly recall that 'd-homotopy relations' require some care (cf. I.2.4, I.2.7). First, we have a *d-homotopy preorder* $f \preceq g$, defined by the existence of a homotopy $f \to g$ (and extending the path-preorder of points); it is consistent with composition but non-symmetric ($f \preceq g$ being equivalent to $Rg \preceq Rf$). Second, we write $f \simeq g$ the equivalence relation generated by $\preceq$: there is a finite sequence $f \preceq f_1 \succeq f_2 \preceq f_3 \ \dots \ g$ (of d-maps between the same objects); it is a congruence of categories on d**Top**.

To conclude this review of Part I, a *d-homotopy equivalence* is a d-map $f: X \to Y$ having a *d-homotopy inverse* $g: Y \to X$, in the sense that $gf \simeq \mathrm{id}X$, $fg \simeq \mathrm{id}Y$; then we say that X and Y have *the same d-homotopy type*, or are *d-homotopy equivalent* (*in n steps* if n instances of the homotopy preorder $\preceq$ are *sufficient* for each of the previous $\simeq$-relations, cf. I.2.7). In particular, if $\mathrm{id}X \preceq gf$ and $\mathrm{id}Y \preceq fg$, we say that X and Y are *immediately* d-homotopy equivalent, *in the future*; if, further, $\mathrm{id}X = gf$, then f embeds X as a *future deformation retract* of Y.

1.2. Homotopy pushouts. Let $f: X \to Y$ and $g: X \to Z$ be two morphisms of directed spaces, with the same domain. The *standard* (directed) *homotopy pushout*, or *h-pushout, from f to g* is a four-tuple $(A; u, v; \lambda)$, as in the left diagram, where $\lambda: uf \to vg: X \to A$ is a homotopy satisfying the following universal property (of *cocomma squares*)

(5)

- for every $\lambda': u'f \to v'g: X \to A'$, there is precisely one map $h: A \to A'$ such that $u' = hu, v' = hv, \lambda' = h \circ \lambda$.

The existence of the solution is proved below; its uniqueness up to isomorphism is obvious. The object A, a 'double mapping cylinder', will be denoted as $\uparrow\! I(f, g)$. Note that $\uparrow\! I(g, f) = R(\uparrow\! I(Rf, Rg))$, where $RX = X^{\mathrm{op}}$ is the reflected d-space (1.1).

As shown in the right diagram, the cylinder itself $\uparrow\! I(X) = X \times \uparrow\! \mathbf{I}$, equipped with the obvious homotopy (*cylinder evaluation*, represented by the identity of the cylinder)

$$\mathrm{ev}^X: \partial^- \to \partial^+: X \to \uparrow\! IX, \qquad \mathrm{ev}^X(x, t) = (x, t), \tag{6}$$

is the h-pushout of the pair $(\mathrm{id}X, \mathrm{id}X)$: by the very definition of homotopies, it establishes a bijection between maps $h: \uparrow\! IX \to W$ and homotopies $h \circ \mathrm{ev}^X: h\partial^- \to h\partial^+: X \to W$. On the other hand, every homotopy pushout in d**Top** can be constructed from the cylinder and *ordinary* pushouts, by the colimit of the following diagram (which amounts to two ordinary pushouts)

$$Y \xleftarrow{\ f\ } X \xrightarrow{\ \partial^-\ } \uparrow\! IX \xleftarrow{\ \partial^+\ } X \xrightarrow{\ g\ } Z \tag{7}$$

i.e. as the quotient of the sum $(\uparrow IX) + Y + Z$, under the equivalence relation identifying $(x, 0)$ with $f(x)$ and $(x, 1)$ with $g(x)$. The forgetful functor $U\colon \mathrm{d}\mathbf{Top} \to \mathbf{Top}$ preserves cylinders and pushouts, hence h-pushouts as well.

1.3. Homotopy pullbacks. Dualising 1.2, the *h-pullback from* $f\colon Y \to X$ *to* $g\colon Z \to X$ is a four-tuple $(A; u, v; \lambda)$, as in the left diagram, where $\lambda\colon fu \to gv\colon A \to X$ satisfies the following universal property (of *comma squares*)

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle u}\uparrow & \overset{\lambda}{\searrow} & \uparrow{\scriptstyle g} \\ A & \xrightarrow[v]{} & Z \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ {\scriptstyle \partial^-}\uparrow & \overset{\mathrm{ev}_X}{\searrow} & \uparrow{\scriptstyle \mathrm{id}} \\ \uparrow PX & \xrightarrow[\partial^+]{} & X \end{array} \tag{8}$$

- for every $\lambda'\colon fu' \to gv'\colon A' \to X$, there is exactly one map $h\colon A' \to A$ such that $u' = uh$, $v' = vh$, $\lambda' = \lambda\circ h$.

The object A will be denoted as $\uparrow P(f, g)$. Again, $\uparrow P(g, f) = R(\uparrow P(Rf, Rg))$. As shown in the right diagram above, the path-object $\uparrow PX$ is the h-pullback of the pair $(\mathrm{id}X, \mathrm{id}X)$, via the obvious homotopy $\mathrm{ev}_X\colon \partial^- \to \partial^+\colon \uparrow PX \to X$ (*path evaluation*, represented by $\mathrm{id}(\uparrow PX)$). All homotopy pullbacks in $\mathrm{d}\mathbf{Top}$ can be constructed from paths and ordinary pullbacks, by the following limit (which amounts to two ordinary pullbacks)

$$Y \xrightarrow{f} X \xleftarrow{\partial^-} \uparrow PX \xrightarrow{\partial^+} X \xleftarrow{g} Z \tag{9}$$

i.e. as the following d-subspace of the product $(\uparrow PX)\times Y \times Z$

$$\uparrow P(f, g) = \{(a, y, z) \in (\uparrow PX)\times Y\times Z \mid a(0) = f(y),\ a(1) = g(z)\}. \tag{10}$$

Note that the forgetful functor $U\colon \mathrm{d}\mathbf{Top} \to \mathbf{Top}$ does *not* preserve path-objects (nor h-pullbacks): $U(\uparrow PX)$ is a subspace of $P(UX)$, and a proper one unless $\uparrow PX$ is just a 'space'.

In the rest of this section we shall study h-pushouts, in a rather formal way (which could be easily extended to an abstract IP-homotopical category with concatenation and 'accelerations', cf. I.2.6.3)); also the dual properties, for h-pullbacks, hold.

1.4. Theorem. [The higher property] *The h-pushout* $A = \uparrow I(f, g)$ *satisfies also a 2-dimensional universal property. Precisely, given two maps* a, b, *two homotopies* σ, τ *and a double homotopy* Φ (I.2.5) *with the following boundaries*

$$\begin{array}{ccccccc} & & Y & & \overset{au}{\underset{bu}{\overset{\downarrow\sigma}{\rightrightarrows}}} & & W \\ & {\scriptstyle f}\nearrow & & \searrow{\scriptstyle u} & & & \| \\ X & & {\scriptstyle\lambda}\downarrow & & A \overset{a}{\underset{b}{\rightrightarrows}} & & W \\ & {\scriptstyle g}\searrow & & \nearrow{\scriptstyle v} & & & \| \\ & & Z & & \overset{av}{\underset{bv}{\overset{\downarrow\tau}{\rightrightarrows}}} & & W \end{array} \qquad \begin{array}{ccc} auf & \xrightarrow{a\lambda} & avg \\ {\scriptstyle \sigma f}\downarrow & \Phi & \downarrow{\scriptstyle \tau g} \\ buf & \xrightarrow[b\lambda]{} & bvg \end{array} \tag{11}$$

$$a, b\colon A \to W, \qquad \sigma\colon au \to bu, \qquad \tau\colon av \to bv, \qquad \Phi\colon X \times {\uparrow}\mathbf{I}^2 \to W,$$
$$\partial_1^-(\Phi) = \Phi.(X \times \partial^- \times {\uparrow}\mathbf{I}) = \sigma \circ f, \qquad \partial_2^-(\Phi) = \Phi.(X \times {\uparrow}\mathbf{I} \times \partial^-) = a \circ \lambda, \ldots$$

there is some homotopy $\varphi\colon a \to b$ such that $\varphi \circ u = \sigma$, $\varphi \circ v = \tau$ (and precisely one which also satisfies $\varphi.(\lambda \times {\uparrow}\mathbf{I}) = \Phi$)).

Proof. By the adjunction ${\uparrow}I \dashv {\uparrow}P$, we can view the data as d-maps with values in ${\uparrow}PW$, namely

$$\begin{array}{ll} \sigma'\colon Y \to {\uparrow}PW, & \tau'\colon Z \to {\uparrow}PW, \\ \Phi' : X \times {\uparrow}\mathbf{I} \to {\uparrow}PW, & \partial^-\Phi' = \sigma' f, \;\; \partial^+\Phi' = \tau' g. \end{array} \tag{12}$$

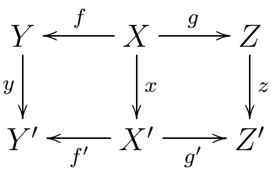
There is thus one map $\varphi'\colon A \to {\uparrow}PW$ such that $\varphi' u = \sigma', \varphi' v = \tau', \varphi' \circ \lambda = \Phi'$. This is the same as a homotopy $\varphi\colon {\uparrow}IA \to W$ satisfying our conditions. Moreover, its lower face is a (and the upper one is b) because

$$\begin{array}{ll} (\partial^-\varphi').u = \partial^-\sigma' = au, & (\partial^-\varphi').v = \partial^-\tau' = av, \\ (\partial^-\varphi') \circ \lambda = \Phi.\partial^-(X \times {\uparrow}\mathbf{I}) = a \circ \lambda. & \end{array} \tag{13}$$

■

1.5. Theorem. [The h-pushout functor] *The double mapping cylinder ${\uparrow}I(f, g)$ acts functorially on the variables f, g (precisely, it is a functor* d**Top**$^{\mathbf{v}}$ $\to$ d**Top**, *where* **v** *is the category formed by two diverging arrows: $\bullet \leftarrow \bullet \rightarrow \bullet$) and turns* coherent *d-homotopies into d-homotopies, as specified below.*

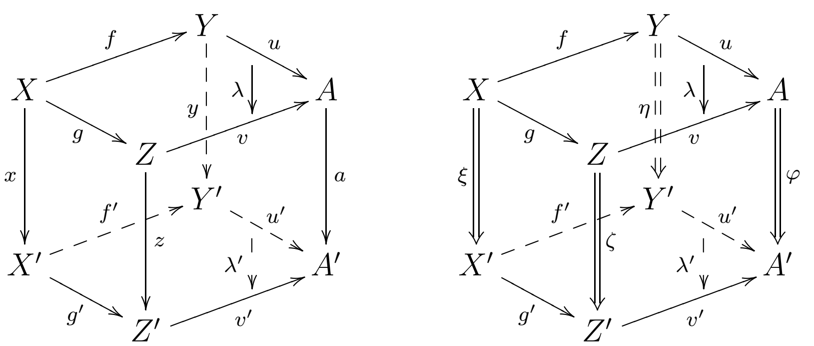
(a) Given a morphism $(x, y, z)\colon (f, g) \to (f', g')$, *consisting of two commutative squares in* d**Top**



(14)

there is precisely one map $a = {\uparrow}I(x, y, z)\colon {\uparrow}I(f, g) \to {\uparrow}I(f', g')$ coherent with the h-pushouts, i.e. such that (as in the left cube below)

$$au = u'y, \qquad av = v'z, \qquad a \circ \lambda = \lambda' \circ x; \tag{15}$$



(16)

(b) Given a coherent system of homotopies*, from the pair* (f, g) *to the pair* (f', g')*, as in the right cube above (where the double arrow at* ξ *stands for* $\xi: x \to x': X \to X'$*; etc).*

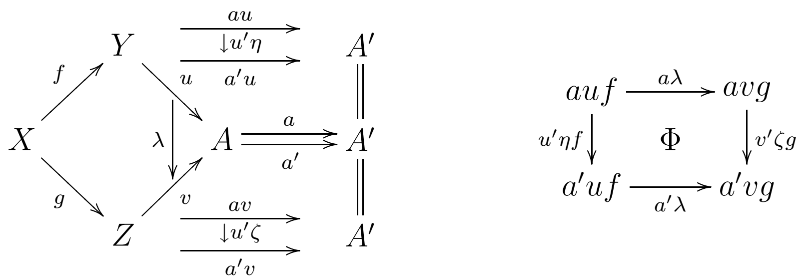
$$\begin{array}{lll} (\xi, \eta, \zeta): (x, y, z) \to (x', y', z'): (f, g) \to (f', g'), & & \\ \xi: x \to x', & \eta: y \to y', & \zeta: z \to z', \\ f'\circ\xi = \eta\circ f, & g'\circ\xi = \zeta\circ g, & \end{array} \tag{17}$$

there is some homotopy $\varphi: a \to a': \uparrow I(f, g) \to \uparrow I(f', g')$ *which completes coherently the cube*

$$\varphi: a \to a', \qquad \varphi\circ u = u'\circ\eta, \qquad \varphi\circ v = v'\circ\zeta, \tag{18}$$

and precisely one such φ *if we also ask that* $\varphi.(\lambda\times\uparrow\mathbf{I}) = \lambda'.(\xi\times\uparrow\mathbf{I}).s$.

Proof. (a) Immediate, from the first universal property of the h-pushout λ of (f, g).
(b) Follows from the 2-dimensional property of the same h-pushout, with respect to the double homotopy $\Phi = \lambda'.(\xi\times\uparrow\mathbf{I}).s: X\times\uparrow\mathbf{I}^2 \to X'\times\uparrow\mathbf{I} \to A'$

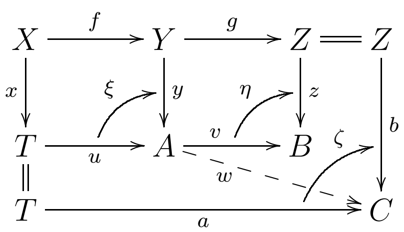


(19)

$$\partial_1^-(\Phi) = u'f'\circ\xi = u'\circ\eta\circ f, \qquad \partial_2^-(\Phi) = \lambda'\circ x = a\circ\lambda, \ldots$$

There is thus some homotopy $\varphi: a \to a'$ such that $\varphi\circ u = u'\circ\eta$, $\varphi\circ v = v'\circ\zeta$; and precisely one which also satisfies $\varphi.(\lambda\times\uparrow\mathbf{I}) = \Phi$. ∎

1.6. Theorem. [Pasting of h-pushouts] *Let* ξ, η, ζ *be standard homotopy pushouts*



(20)

Then B *and* C *are* immediately d-homotopy equivalent, in the future (1.1)*, by canonical maps and homotopies.*

Precisely, define $w: A \to C$, $h: B \to C$ *and* $k: C \to B$ *through the universal property of* ξ, η, ζ*, respectively (and the concatenation of homotopies, for* k*).*

$$
\begin{array}{lll}
w.u = a\colon T \to C, & w.y = bg\colon Y \to C, & w\circ\xi = \zeta\colon ax \to bgf, \\
h.v = w\colon A \to C, & h.z = b\colon Z \to C, & h\circ\eta = 0\colon wy \to bg, \\
k.a = vu\colon T \to B, & k.b = z\colon Z \to B, & k\circ\zeta = (v\circ\xi + \eta\circ f)\colon vux \to zgf.
\end{array} \tag{21}
$$

Then h, k form an immediate homotopy equivalence, with $\mathrm{id}C \preceq hk$ *and* $\mathrm{id}B \preceq kh$*; the following higher coherence relations can also be obtained (note we are* not *saying that $\psi\circ v = 0$).*

$$
\begin{array}{llll}
\varphi\colon \mathrm{id}C \to hk, & \varphi\circ a = 0_a, & \varphi\circ b = 0_b, & \\
\psi\colon \mathrm{id}B \to kh, & \psi\circ z = 0_z, & \psi\circ vu = 0_{vu}, & \psi\circ vy = 0 + \eta.
\end{array} \tag{22}
$$

By reflection, if we paste d-homotopy pushouts (placed as above) *with the opposite direction of homotopies, we get an immediate d-homotopy equivalence* in the past.

Proof. (This result will be essential for the sequel, e.g. to prove the homotopical exactness of the cofibre sequence, in Theorem 2.5. It is a refinement of a similar result in [6], 3.4.)

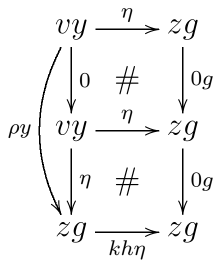
First, the higher universal property of ζ (1.4) yields a homotopy $\varphi\colon \mathrm{id}C \to hk$ (with $\varphi\circ a = 0_a$, $\varphi\circ b = 0_b$), provided by the *acceleration* double homotopy $\zeta \to \zeta + 0$ (I.2.6.3), denoted by #

$$
\begin{array}{ccc}
ax & \xrightarrow{\ \zeta\ } & bgf \\
{\scriptstyle 0_a x}\downarrow & \# & \downarrow{\scriptstyle 0_b gf} \\
hkax & \xrightarrow[hk\zeta]{} & hkbgf
\end{array}
\qquad
\begin{array}{c}
hka = hvu = wu = a, \\
hkb = hz = b, \\
hk\circ\zeta = hv\circ\xi + h\circ\eta\circ f = w\circ\xi + 0 = \zeta + 0.
\end{array} \tag{23}
$$

We exploit now the higher universal property of ξ to link the maps $v, kw\colon A \to B$; consider the following three double homotopies (acceleration, degeneracy and upper connection)

$$
\begin{array}{ccccc}
vux & & \xrightarrow{\quad v\xi \quad} & & vyf \\
{\scriptstyle 0x}\downarrow & & \# & & \downarrow{\scriptstyle 0f} \\
vux & \xrightarrow{v\xi} & vyf & \xrightarrow{0} & vyf \\
{\scriptstyle 0x}\downarrow & \# & \downarrow{\scriptstyle 0} & \# & \downarrow{\scriptstyle \eta f} \\
kwux & \xrightarrow[v\xi]{} & vyf & \xrightarrow[\eta f]{} & kwyf
\end{array}
\ \ {\scriptstyle (0+\eta)f}
\qquad
\begin{array}{c}
kwu = ha = vu, \\[2ex]
kw\circ\xi = k\circ\zeta = v\circ\xi + \eta\circ f.
\end{array} \tag{24}
$$

Their pasting yields a homotopy $\rho\colon v \to kw$ such that $\rho\circ u = 0$, $\rho\circ y = 0 + \eta$. Finally, the higher property of η produces a homotopy $\psi\colon \mathrm{id}B \to kh$ (such that $\psi\circ v = \rho$, $\psi\circ z = 0$), through a double homotopy deriving from degeneracy and lower connection



$$\rho\circ y = 0 + \eta,$$

$$kh\circ\eta = 0\colon kwy \to kbg. \qquad (25)$$

■

1.7. Cofibrations. Say that a d-map $u\colon X \to A$ is a *lower d-cofibration* if (see the left diagram below), for every d-space W and every d-map $h\colon A \to W$, every d-homotopy $\psi\colon h' = hu \to k'$ can be 'extended' to a d-homotopy $\varphi\colon h \to k$ on A (so that $\varphi\circ u = \psi$, whence $ku = k'$)

(26)

The opposite notion (for every k and $\psi\colon h' \to k' = ku$ there is some φ such that $\varphi u = \psi$) is called *upper d-cofibration*; a *bilateral d-cofibration* has to satisfy both conditions.

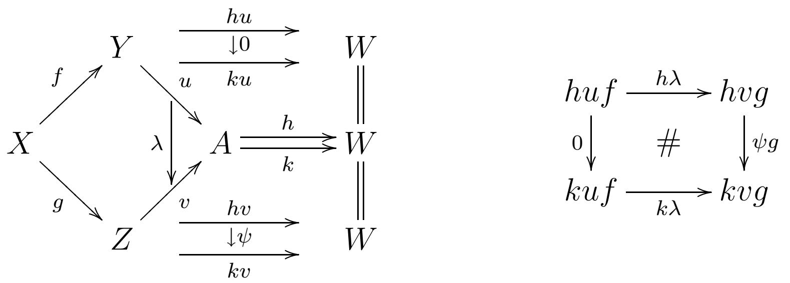
The right diagram above shows the definition of a *lower d-fibration* $p\colon X \to B$: for every $h\colon W \to X$ and $\psi\colon h' = ph \to k'$ there is some $\varphi\colon h \to k$ which lifts y ($p\circ\varphi = \psi$).

1.8. Theorem. *(a) In every h-pushout $A = \uparrow I(f, g)$ (as in (5)), the first 'injection' $u\colon Y \to A$ is an upper d-cofibration, while the second $v\colon Z \to A$ is a lower one.*
(a) In every h-pullback, the first 'projection' is an upper d-fibration, while the second is a lower one.*

Proof. It is sufficient to verify the second statement of (a). Take a d-map $h\colon A \to W$ and a d-homotopy $\psi\colon h' = hv \to k'$. Then, there is one map $k\colon A \to W$ such that

$$ku = hu, \qquad kv = k', \qquad k\circ\lambda = h\circ\lambda + \psi\circ g\colon huf \to k'g\colon X \to W. \qquad (27)$$

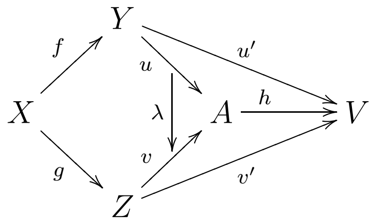
Moreover, by the last relation, we can construct a double homotopy as in the right diagram below (as in (24), by acceleration, degeneracy and upper connection)



(28)

and this produces a homotopy $\varphi\colon h \to k$ such that $\varphi\circ v = \psi$. ■

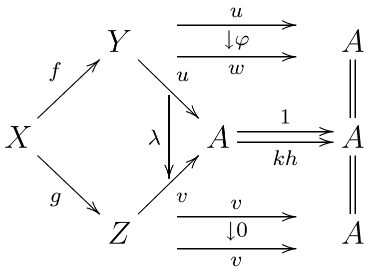
1.9. Theorem. [h-pushouts of cofibrations] *If $f\colon X \to Y$ is a lower d-cofibration and $g\colon X \to Z$ any map, the obvious map $h\colon A \to V$ from the h-pushout $A = \uparrow I(f,g)$ to the* ordinary *pushout V is an immediate d-homotopy equivalence, in the past*

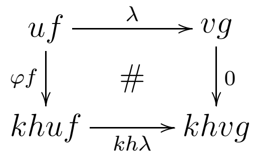


$$hu = u',\ hv = v', \qquad h\circ\lambda = 0\colon u'f \to v'g. \tag{29}$$

Proof. The extension property of f ensures that the homotopy $\lambda\colon uf \to vg$ can be extended to some homotopy $\varphi\colon u \to w\colon Y \to A$; thus $\varphi\circ f = \lambda$ and $wf = vg$. There is then one map $k\colon V \to A$ such that $ku' = w,\ kv' = v$.

First, $1_A \preceq kh$, by the 2-dimensional property of λ, using a lower connection (since $\varphi\circ f = \lambda$ and $kh\circ\lambda = 0$)





(30)

Second, $1_V \preceq hk$, by the 2-dimensional property of the ordinary pullback V (which trivially holds since homotopies are represented by a cocylinder). Since the pair of homotopies $h\circ\varphi\colon u' \to hw,\ 0\colon v' \to v'$ is coherent with f, g ($h\circ\varphi\circ f = h\circ\lambda = 0 = 0\circ g$), there is one homotopy ψ such that $\psi\circ u' = h\circ\varphi$ and $\psi\circ v' = 0_{v'}$; finally, $\psi\colon 1_V \to hk$, since

$$\partial^-(h\circ\varphi) = hu = u', \qquad \partial^+(h\circ\varphi) = hw = hk.u', \qquad \partial^\pm(0_{v'}) = v'.$$ ■

2. Mapping cones and the cofibre sequence

Mapping cones (i.e., homotopy cokernels) and suspensions are particular instances of homotopy pushouts. The cofibre sequence of a map has strong properties of 'homotopical exactness': it is homotopy equivalent to a sequence of iterated mapping cones.

2.1. Mapping cones. In contrast with ordinary homotopy, the lack of a reversion for directed homotopies produces *two* mapping cones, generally non isomorphic, yet *linked by reflection.*

Every d-map $f\colon X \to Y$ has an *upper h-cokernel*, or *upper mapping cone* $\uparrow C^+ f = \uparrow I(f, t_X)$, the h-pushout *from f to* the terminal map $t_X\colon X \to \{*\}$, as in the left diagram below; it can be obtained as the quotient of the sum $\uparrow IX + Y + \{*\}$ under the equivalence relation identifying $(x, 0)$ with $f(x)$ and $(x, 1)$ with $\{*\}$, for all $x \in X$ (cf. (7)); its vertex v^+ 'is in the future': it can be reached from every point (it is a maximum in the path preorder, and actually the only one)

$$\begin{array}{ccc} X & \xrightarrow{t_X} & \{*\} \\ {\scriptstyle f}\downarrow & \nearrow{\scriptstyle \gamma} & \downarrow{\scriptstyle v^+} \\ Y & \xrightarrow[c^-]{} & \uparrow C^+f \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle t_X}\downarrow & \nearrow{\scriptstyle \gamma} & \downarrow{\scriptstyle c^+} \\ \{*\} & \xrightarrow[v^-]{} & \uparrow C^-f \end{array} \qquad (31)$$

(This is called the 'lower' mapping cone in [7]; the present choice of terms, based on the vertex rather than the basis, derives from the analysis of contractility, at the end of 2.2).

Symmetrically, one obtains the *lower mapping cone* $\uparrow C^-f = \uparrow I(t_X, f) = R\uparrow C^+(Rf))$, as in the right diagram; the equivalence relation identifies now $(x, 0)$ with $*$ and $(x, 1)$ with $f(x)$; the vertex v^- 'is in the past': it can reach every point (is a minimum in the path preorder).

As an easy consequence of the homotopy invariance of the h-pushout functor (Theorem 1.5), the mapping cone functor is d-homotopy invariant as well, in the same sense

$$\uparrow C^\alpha \colon \mathrm{d}\mathbf{Top}^{\mathbf{2}} \to \mathrm{d}\mathbf{Top}; \qquad (32)$$

($\mathbf{2}$ is 'the one-arrow category' $\bullet \to \bullet$; an object of $\mathrm{d}\mathbf{Top}^{\mathbf{2}}$ is a d-map, while a morphism is a commutative square of d-maps, as in (14) with $Z = \{*\}$).

Note that h-cokernels are based on the terminal object $\{*\}$. Working dually with the initial object $\emptyset$ would give trivial results: all h-kernels would be empty; as in the ordinary case, one has to move to the *pointed* case to get h-kernels of interest; this will be considered in the next section.

2.2. Cones. Applying these constructs to an identity, we have the upper d-cone $\uparrow C^+X = \uparrow C^+(\mathrm{id}X)$ of an object, from the basis (in the past) to the vertex (in the future); and the lower one, $\uparrow C^-X$. The functors $\uparrow C^\alpha \colon \mathrm{d}\mathbf{Top} \to \mathrm{d}\mathbf{Top}$ are d-homotopy invariant.

In $\mathbf{Top}$, the cone of the circle is the compact disc. In $\mathrm{d}\mathbf{Top}$, we get six different d-spaces by letting $\uparrow C^\alpha$ act on $\mathbf{S}^1$, $\uparrow\mathbf{S}^1$ and $\uparrow\mathbf{O}^1$. Thus, the natural circle $\mathbf{S}^1$ has an *upper* d-cone $\uparrow C^+\mathbf{S}^1$, where a path has to move - anyhow - towards the centre (the vertex), at least in the weak sense, and a *lower* d-cone $\uparrow C^-\mathbf{S}^1 = R(\uparrow C^+\mathbf{S}^1)$, where paths proceed the other way and the centre is the only point which can reach all the others. In the d-spaces $\uparrow C^\alpha(\uparrow\mathbf{S}^1)$, a 'pointlike vortex' appears at the centre (showing that such d-spaces cannot be defined by a local preorder, cf. I.1.6);

$\uparrow C^+\mathbf{S}^1 \qquad \uparrow C^-\mathbf{S}^1 \qquad \uparrow C^+\uparrow\mathbf{S}^1 \qquad \uparrow C^-\uparrow\mathbf{S}^1$ (33)

the fundamental category of these d-spaces has been computed in I.3.5, I.3.7.

In **Top**, again, the projective plane $\mathbf{P}^2$ is the ordinary mapping cone of the endomap $f\colon \mathbf{S}^1 \to \mathbf{S}^1$ of degree 2. In d**Top**, the same mapping, viewed as $f\colon \mathbf{S}^1 \to \mathbf{S}^1$ and $g\colon {\uparrow}\mathbf{S}^1 \to {\uparrow}\mathbf{S}^1$, yields four directed versions of the projective plane: the quotients of the previous disks under the usual equivalence relation, namely ${\uparrow}C^+f$, ${\uparrow}C^-f \cong R({\uparrow}C^+f)$, ${\uparrow}C^+g$, ${\uparrow}C^-g \cong R({\uparrow}C^+g)$. The last isomorphism follows from ${\uparrow}\mathbf{S}^1$ being reflexive, i.e. isomorphic to $R({\uparrow}\mathbf{S}^1)$.

A d-space X is said to be *future contractible* if it has a future deformation retract at some point (1.1; I.2.7). This happens if and only if the basis $c^-\colon X \to {\uparrow}C^+X$ has a retraction $h\colon {\uparrow}C^+X \to X$ (apply the universal property of ${\uparrow}C^+X$). The cone ${\uparrow}C^+X$ itself is future contractible (to its vertex v^+, and only to this point) by means of the homotopy induced on ${\uparrow}C^+X \times {\uparrow}\mathbf{I}$ by the lower connection (constant at $t = 1$ and at $t' = 1$)

$$\mathbf{g}^-\colon {\uparrow}C^+X \times {\uparrow}\mathbf{I} \to {\uparrow}C^+X, \qquad \mathbf{g}^-[x, t, t'] = [x, \max(t, t')]. \tag{34}$$

2.3. Suspension. The (directed, non-pointed) *suspension* ${\uparrow}\Sigma X$ is a lower and upper d-cone, at the same time

$$\begin{array}{ccc} X & \xrightarrow{t_X} & \{*\} \\ {\scriptstyle t_X}\downarrow & \overset{\gamma}{\nearrow} & \downarrow{\scriptstyle v^+} \\ \{*\} & \xrightarrow[v^-]{} & {\uparrow}\Sigma X \end{array} \qquad \begin{array}{c} {\uparrow}\Sigma X = {\uparrow}I(t_X, t_X) = {\uparrow}C^-(t_X) = {\uparrow}C^+(t_X), \\ R.{\uparrow}\Sigma = {\uparrow}\Sigma.R. \end{array} \tag{35}$$

Concretely, it is the quotient of the sum ${\uparrow}IX + \{v^-\} + \{v^+\}$ which identifies the lower basis with a *lower vertex* v^-, and the upper basis with an *upper vertex* v^+. It is equipped with a homotopy (*suspension evaluation*)

$$\mathrm{ev}^X\colon v^-t_X \to v^+t_X\colon X \to {\uparrow}\Sigma X, \qquad (x, t) \mapsto [x, t], \tag{36}$$

which is universal for homotopies between constant maps. In particular, ${\uparrow}\Sigma(\{*\}) = {\uparrow}\mathbf{I}$ and ${\uparrow}\Sigma(\emptyset) = \mathbf{S}^0$ (note that the latter is not a quotient of ${\uparrow}I(\emptyset) = \emptyset$!).

The suspension ${\uparrow}\Sigma$ is an endofunctor of d**Top** (by 1.5): given $f\colon X \to Y$, the suspended map ${\uparrow}\Sigma f\colon {\uparrow}\Sigma X \to {\uparrow}\Sigma Y$ is the unique morphism which satisfies the conditions

$${\uparrow}\Sigma f.v^- = v^-, \qquad {\uparrow}\Sigma f.v^+ = v^+, \qquad ({\uparrow}\Sigma f)\circ\mathrm{ev}^X = \mathrm{ev}^Y\circ f; \tag{37}$$

moreover, by 1.5, *this functor is homotopy invariant*: given a homotopy $\varphi\colon f \to g$, there is some homotopy $\psi\colon {\uparrow}\Sigma f \to {\uparrow}\Sigma g$ (in fact, there is precisely one such that $\psi.(\mathrm{ev}^X \times {\uparrow}\mathbf{I}) = \mathrm{ev}^Y.(\varphi \times {\uparrow}\mathbf{I}).s$). Therefore, ${\uparrow}\Sigma$ preserves immediate homotopy equivalences, and n-step homotopy equivalences as well.

The (unpointed!) suspension of $\mathbf{S}^0$ is the quotient of ${\uparrow}\mathbf{I} + {\uparrow}\mathbf{I}$ which identifies lower and upper endpoints, separately. This coincides with the d-structure induced by $\mathbf{R}\times{\uparrow}\mathbf{R}$ on the standard circle (or any circle), called the *ordered circle* (in I.1.2.5), because it is of (partial) order type

$$ {\uparrow}\Sigma(\mathbf{S}^0) = {\uparrow}\mathbf{O}^1 \subset \mathbf{R}\times{\uparrow}\mathbf{R}. \tag{38} $$

More generally, one can define the *ordered n-sphere* ${\uparrow}\mathbf{O}^n = {\uparrow}\Sigma^n(\mathbf{S}^0)$. It is isomorphic to the structure induced on the standard n-sphere by $\mathbf{R}\times{\uparrow}\mathbf{R}^n$, as well as to the pasting of two ordered discs ${\uparrow}\mathbf{B}^n \subset {\uparrow}\mathbf{R}^n$ along their boundary; the latter description shows that ${\uparrow}\mathbf{O}^n$ is indeed of order type (while $\mathbf{R}\times{\uparrow}\mathbf{R}^n$ is just of preorder type, since the natural $\mathbf{R}$ has the chaotic preorder, cf. 1.1).

2.4. The cofibre sequence. Every d-map $f\colon X \to Y$ has a *lower cofibre sequence*, produced by lower h-cokernels (as well as an *upper* one)

$$ X \xrightarrow{f} Y \xrightarrow{x} {\uparrow}C^-f \xrightarrow{d} {\uparrow}\Sigma X \xrightarrow{{\uparrow}\Sigma f} {\uparrow}\Sigma Y \xrightarrow{{\uparrow}\Sigma x} {\uparrow}\Sigma({\uparrow}C^-f) \xrightarrow{{\uparrow}\Sigma d} {\uparrow}\Sigma^2 X \dots \tag{39} $$

$$ \begin{array}{ll} x = c^+\colon Y \to {\uparrow}C^-f; & \\ d.v^- = v^-\colon \{*\} \to {\uparrow}\Sigma X, & d.x = v^+.t_Y\colon Y \to {\uparrow}\Sigma X, \\ d\circ\gamma = \mathrm{ev}^X\colon v^-.t_X \to v^+.t_X\colon X \to {\uparrow}\Sigma X. & \end{array} $$

(More generally, this works for any category with cylinder functor, terminal object and pushouts; cf. [7], Section 1.) Moreover, as sketched below, in 2.5 (and proved - more generally - in [7], 1.10, 3.4), this sequence can be linked, via a *homotopically commutative diagram*, to a sequence of iterated h-cokernels of f, where each map is, *alternatively*, the *lower or upper* h-cokernel of the preceding one

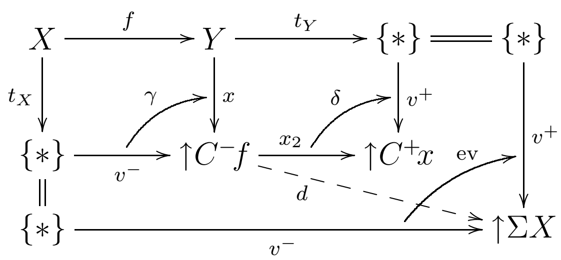
$$ \begin{array}{ccccccccccccccc} X & \xrightarrow{f} & Y & \xrightarrow{x} & {\uparrow}C^-f & \xrightarrow{d} & {\uparrow}\Sigma X & \xrightarrow{{\uparrow}\Sigma f} & {\uparrow}\Sigma Y & \xrightarrow{{\uparrow}\Sigma x} & {\uparrow}\Sigma({\uparrow}C^-f) & \xrightarrow{{\uparrow}\Sigma d} & {\uparrow}\Sigma^2 X \dots \\ \| & & \| & & \| & & \uparrow h_1 & \simeq & \uparrow h_2 & \simeq & \uparrow h_3 & \simeq & \uparrow h_4 \\ X & \xrightarrow[f]{} & Y & \xrightarrow[x]{} & {\uparrow}C^-f & \xrightarrow[x_2]{} & {\uparrow}C^+x & \xrightarrow[x_3]{} & {\uparrow}C^-x_2 & \xrightarrow[x_4]{} & {\uparrow}C^+x_3 & \xrightarrow[x_5]{} & {\uparrow}C^-x_4 \dots \end{array} \tag{40} $$

If f is an upper d-cofibration (1.7), we can replace ${\uparrow}C^-f$ with the ordinary pushout of f along $X \to \{*\}$, i.e. the quotient $Y/f(X)$ (Theorem 1.9), which is immediately d-homotopy equivalent to the former (in the future).

2.5. Theorem. [Homotopical exactness of the cofibre sequence] *Every comparison* h_n *in the diagram* (40) *is a d-homotopy equivalence in* m *steps (possibly less), where* m *is the integral part of* $(n+2)/3$.

Proof. (Note that this need not be true in the setting of [7], where the concatenation of homotopies is missing; and actually, it is not true for cochain algebras, where the comparisons are just homotopy equivalences of cochain complexes.)

In fact, the map $h_1\colon {\uparrow}C^+x \to {\uparrow}\Sigma X$ derives from pasting the h-pushouts ${\uparrow}C^-f$, ${\uparrow}C^+x$ and comparing them to the suspension (it is constructed as h in (21)); Theorem 1.6 ensures that it is an immediate d-homotopy equivalence, *in the future*



(41)

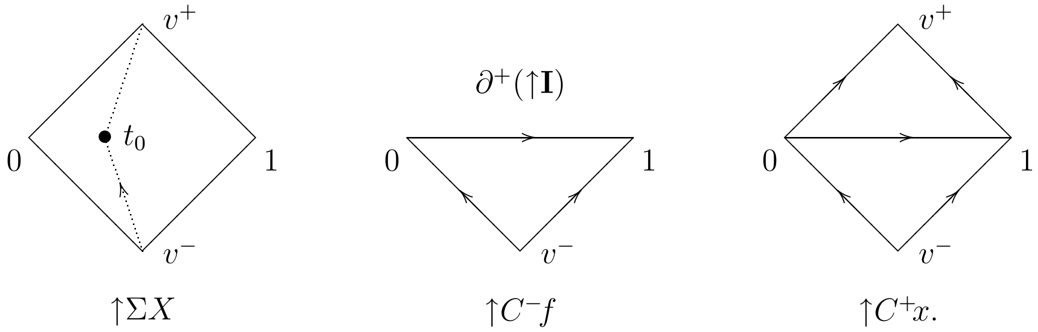
$$h_1.x_2 = d\colon {\uparrow}C^-f \to {\uparrow}\Sigma X, \qquad h_1.v^+ = v^+\colon \{*\} \to {\uparrow}\Sigma X,$$
$$h_1.\delta = 0\colon dx \to v^+.t_Y\colon Y \to {\uparrow}\Sigma X.$$

The maps h_2 and h_3 are similarly defined, and are immediate d-homotopy equivalences as well, *in the past and the future*, respectively. Then $h_4 = {\uparrow}\Sigma h_1.k_4$ is a 2-step d-homotopy equivalence, obtained by composing ${\uparrow}\Sigma h_1$ (an equivalence in the future, by the homotopy invariance of suspension, in 2.3) with a map $k_4\colon {\uparrow}C^-x_4 \to {\uparrow}\Sigma({\uparrow}C^+x)$ obtained as the previous three comparisons (an equivalence in the past). One proceeds this way, adding one step every third index.

Finally, by the homotopy invariance of the mapping cone functors (32), the cofibre sequence acts functorially on the variable f (in d$\mathbf{Top}^2$), and this functor preserves d-homotopies. ■

2.6. An example. Let $X = c_0\mathbf{I}$ be the Euclidean interval with the d-discrete structure (where d-paths are constant) and $Y = {\uparrow}\mathbf{I}$; let $f\colon X \to Y$ be the d-map provided by id$\mathbf{I}$. We want to compute the comparison $h_1\colon {\uparrow}C^+x \to {\uparrow}\Sigma X$.

First, ${\uparrow}\Sigma X$ is the quotient of $c_0\mathbf{I}\times{\uparrow}\mathbf{I}$ which identifies the lower (resp. upper) basis with a point v^- (resp. v^+); its d-paths $a\colon v^- \to v^+$ *move up the fibres* (more precisely, they are all maps $t \mapsto [t_0, a(t)]$ where $a\colon {\uparrow}\mathbf{I} \to {\uparrow}\mathbf{I}$ is a d-path from 0 to 1), and are homotopic (with fixed endpoints) if and only if they pertain to the same fibre (same $t_0 \in c_0\mathbf{I}$); thus, the set of homotopy classes ${\uparrow}\Pi_1({\uparrow}\Sigma X)(v^-, v^+)$ can be identified with the interval $[0,1]$



(42)

On the other hand, $\uparrow C^+ x = \uparrow C^+(\partial^+ \colon Y \to \uparrow C^- f)$ has a coarser structure on its 'upper half', deriving from the cone on $Y = \uparrow\mathbf{I}$. The comparison $h_1 \colon \uparrow C^+ x \to \uparrow\Sigma X$ is now the obvious map which collapses this upper half.

3. Pointed d-spaces and the fibre sequence

Pointed d-spaces have a homotopy structure similar to the unpointed case. Their homotopy fibres (i.e., homotopy kernels) and loop-objects form a fibre sequence with exactness properties dual to the ones of the cofibre sequence. But the derived sequence of higher homotopy monoids is not exact.

3.1. Pointed d-spaces. The category d$\mathbf{Top}_*$ of *pointed d-spaces* behaves (with respect to d$\mathbf{Top}$) much as its ordinary counterpart, the category $\mathbf{Top}_*$ of pointed spaces (with respect to $\mathbf{Top}$).

Limits and coequalisers in d$\mathbf{Top}_*$ are computed as in d$\mathbf{Top}$ (and conveniently pointed); the sum is a quotient of the unpointed sum, modulo the equivalence relation which identifies all base points; but it will be preferably realised in the usual form of a *join*, as a pointed subspace of the corresponding product. Thus, in the binary case

$$(X, x_0) \vee (Y, y_0) = ((X \times \{y_0\}) \cup (\{x_0\} \times Y), (x_0, y_0)) \subset (X \times Y, (x_0, y_0)). \qquad (43)$$

Also here, the smash product, formed by collapsing the join in the product

$$(X, x_0) \wedge (Y, y_0) = ((X \times Y)/((X \times \{y_0\}) \cup (\{x_0\} \times Y)), [(x_0, y_0)]), \qquad (44)$$

yields a symmetric monoidal structure, whose identity is $(\mathbf{S}^0, 1)$.

A (pointed directed) homotopy of pointed d-maps, $\varphi \colon f \to g \colon (X, x_0) \to (Y, y_0)$ is defined as a d$\mathbf{Top}$-homotopy such that $\varphi(x_0, t) = y_0$ (for all t). To represent such homotopies, the (pointed directed) *cylinder* of (X, x_0) is a quotient of the cylinder of X, where the fibre at the base point, $\{x_0\} \times \uparrow\mathbf{I}$, is collapsed to a base point

$$\uparrow I(X, x_0) = (X \times \uparrow\mathbf{I})/(\{x_0\} \times \uparrow\mathbf{I}) = (X, x_0) \wedge \uparrow\mathbf{I}_*, \qquad (45)$$

and can be obtained as the smash product with the *pointed d-interval* $\uparrow\mathbf{I}_*$, where an isolated base-point has been added (applying the left adjoint $(-)_*\colon \mathrm{d}\mathbf{Top} \to \mathrm{d}\mathbf{Top}_*$ of the forgetful functor).

More simply, the *pointed path functor* is the same as the unpointed one, pointed at the constant path at the base point

$$\uparrow P(X, x) = (\uparrow P(X), 0_x), \qquad \uparrow I \dashv \uparrow P \qquad (\text{on } \mathrm{d}\mathbf{Top}_*). \tag{46}$$

3.2. Pointed h-pushouts. All pointed homotopy pushouts can be constructed from the pointed cylinder described above, via the usual colimit (7) calculated in $\mathrm{d}\mathbf{Top}_*$. Thus, the *pointed d-suspension* can be obtained as a quotient of the unpointed cylinder

$$\uparrow\Sigma(X, x_0) = (X \times \uparrow\mathbf{I})/((\{x_0\} \times \uparrow\mathbf{I}) \cup (X \times \mathbf{S}^0)) = (X, x_0) \wedge (\uparrow\mathbf{S}^1, [0]), \tag{47}$$

and also as the smash product with the directed circle $\uparrow\mathbf{S}^1 = \uparrow\mathbf{I}/\partial\mathbf{I}$ pointed at $[0] = [1]$.

As a relevant difference from the ordinary case, *the pointed d-suspension of a d-sphere is different from the unpointed one* (cf. (38)): here we obtain $\uparrow\Sigma(\mathbf{S}^0, 1) = (\uparrow\mathbf{S}^1, [0])$. Actually, it is easy to deduce from the definition of higher directed spheres, $\uparrow\mathbf{S}^n = (\uparrow\mathbf{I}^n)/(\partial\mathbf{I}^n)$ (I.1.2.4), that

$$\uparrow\Sigma^n(\mathbf{S}^0, 1) = (\uparrow\mathbf{S}^n, [0]). \tag{48}$$

The cofibre sequence of a *pointed* d-map $f\colon X \to Y$ behaves as in the unpointed case (Section 2).

3.3. Pointed h-pullbacks. When a pointed object is denoted by a single letter, say X, its base point will be written as $*_X$ or $*$.

All pointed h-pullbacks can be constructed as pointed limits (cf. (9)), from the pointed cocylinder (46). Dualising 2.1, the *upper homotopy kernel* of a pointed d-map $f\colon X \to Y$, or *upper homotopy fibre* $\uparrow K^+f = \uparrow P(f, i_Y)$, is the h-pullback *from* f *to* the initial map $i_Y\colon \{*\} \to Y$, as in the left diagram below; similarly one has the *upper cocone* of an object, $\uparrow K^+X$, as in the right diagram

(49)

$$\uparrow K^+f = \uparrow P(f, i_Y) = \{(x, b) \in X \times \uparrow PY \mid b(0) = f(x), b(1) = *_Y\},$$
$$\uparrow K^+X = \uparrow K^+(\mathrm{id}X) = \{a \in \uparrow PX \mid a(1) = *_X\},$$

pointed at $(*_X, 0_{*_Y})$ and 0_{*_X}, respectively. The lower analogues are written $\uparrow K^-f$ and $\uparrow K^-X$.

Dualising 2.3, the (directed) *loop-object* $\uparrow\Omega X$ is a lower and upper h-kernel, at the same time

$$\begin{array}{ccc} \{*\} & \xrightarrow{f} & X \\ \uparrow & \overset{\mathrm{ev}_X}{\Rightarrow} & \uparrow \\ \uparrow\Omega X & \longrightarrow & \{*\} \end{array} \qquad (50)$$

$$\uparrow\Omega X = \uparrow P(i_X, i_X) = \{c \in \uparrow PX \mid c(0) = *_X = c(1)\}, \qquad * = 0_{*_X}.$$

It comes equipped with a d-homotopy $ev_X\colon 0 \to 0\colon \uparrow\Omega X \to X$ (*loop evaluation*), universal for d-endohomotopies of null maps in X: for every $\varphi\colon 0 \to 0\colon A \to X$ there is one d-map $f\colon A \to \uparrow\Omega X$ such that $\varphi = \mathrm{ev}_X \circ f$. The functors produced by h-kernel, cocone and loop-objects (on d$\mathbf{Top}_*^\mathbf{2}$ and d$\mathbf{Top}_*$) are homotopy invariant, as proved previously for the dual cases.

A pointed d-map $f\colon X \to Y$ has a *lower fibre sequence*, produced by lower h-kernels

$$\ldots \uparrow\Omega^2 Y \xrightarrow{\uparrow\Omega d} \uparrow\Omega(\uparrow K^- f) \xrightarrow{\uparrow\Omega p} \uparrow\Omega X \xrightarrow{\uparrow\Omega f} \uparrow\Omega Y \xrightarrow{d} \uparrow K^- f \xrightarrow{p} X \xrightarrow{f} Y \qquad (51)$$

and satisfying dual properties with respect to the cofibre sequence (Section 2). In particular, it is d-homotopy equivalent to the sequence of iterated lower and upper h-kernels; and acts functorially on the variable f, preserving (pointed) d-homotopies of d-maps. If f is a (pointed) upper d-fibration (cf. 1.7), we can replace $\uparrow K^- f$ with the ordinary kernel of f, the fibre $F = f^{-1}\{*_Y\}$, which is d-homotopy equivalent.

3.4. Higher homotopy monoids. A pointed d-space A produces a sequence of d-homotopy invariant functors ($n \geq 0$)

$$\uparrow\pi_n^A\colon \mathrm{d}\mathbf{Top}_* \to \mathbf{Set}_*, \qquad \uparrow\pi_n^A(X) = [\uparrow\Sigma^n A, X]_* = [A, \uparrow\Omega^n X]_*, \qquad (52)$$

where $[A, X]_*$ is the pointed set of classes of maps $A \to X$ in d$\mathbf{Top}_*$, up to pointed d-homotopy. Plainly, the concatenation of loops $(\uparrow\Omega X) \times (\uparrow\Omega X) \to \uparrow\Omega X$ produces a structure of monoid on all such sets, for $n \geq 1$, which is commutative for $n \geq 2$ (by the middle-four interchange of pasting, as usual; cf. I.2.5.3).

The superscript A will be omitted when $A = \mathbf{S}^0 = \{-1, 1\}$, pointed at 1. Then, $\uparrow\pi_0(X)$ is the set of d-path components $\uparrow\Pi_0(X) = |X|/\simeq$ (I.1.5.2), pointed at the component of the base point, while the remaining functors $\uparrow\pi_n$ yield the *homotopy monoids* of a pointed d-space

$$\begin{array}{ll} \uparrow\pi_0(X) = [\uparrow\mathbf{S}^0, X]_* = (|X|/\simeq, [*_X]), & \\ \uparrow\pi_n(X) = [\uparrow\mathbf{S}^n, X]_* = [\mathbf{S}^0, \uparrow\Omega^n X]_* = \uparrow\pi_0(\uparrow\Omega^n X) & (n \geq 1). \end{array} \qquad (53)$$

In particular, $\uparrow\pi_1(X)$ is (as previously defined, in I.3.3) the monoid of endoarrows $[c]\colon *_X \to *_X$ in the fundamental category $\uparrow\Pi_1(X)$

$$\uparrow\pi_1(X) = [\uparrow\mathbf{S}^1, X]_* = \uparrow\pi_0(\uparrow\Omega X) = \uparrow\Pi_1(X)(*, *); \qquad (54)$$

it is (strictly) invariant up to pointed d-homotopy and can be computed via $\uparrow\Pi_1(X)$ (by the methods developed in I.3, especially the van Kampen-type pasting theorem).

Replacing $\uparrow\mathbf{S}^1$ with the reversible circle $\tilde{\mathbf{I}}/\partial\mathbf{I}$ (I.1.3c), we get the group of invertible loops $\mathrm{Inv}(\uparrow\pi_1(X))$. The ordered circle $\uparrow\mathbf{O}^1$ (cf. (38)), pointed at its minimum or maximum, yields two other d-homotopy invariants of interest, $[(\uparrow\mathbf{O}^1, v^\alpha), -]_*\colon \mathrm{d}\mathbf{Top}_* \to \mathbf{Set}_*$.

3.5. Sequences of homotopy monoids. Applying the functor $\uparrow\pi_0$ to the fibre sequence of a pointed d-map $f\colon X \to Y$ (see (51)), we obtain the *fibre sequence of homotopy monoids* (and pointed sets)

$$\begin{aligned} \cdots \xrightarrow{f_{*2}} \uparrow\pi_2 Y \xrightarrow{d_{*1}} \uparrow\pi_1(\uparrow K^- f) \xrightarrow{p_{*1}} \uparrow\pi_1 X \xrightarrow{f_{*1}} \uparrow\pi_1 Y \\ \xrightarrow{d_{*0}} \uparrow\pi_0(\uparrow K^- f) \xrightarrow{p_{*0}} \uparrow\pi_0 X \xrightarrow{f_{*0}} \uparrow\pi_0 Y \end{aligned} \tag{55}$$

which is of order two but *not* exact, and probably much less interesting than its ordinary analogue, the fibre sequence of homotopy groups (which *is* exact).

Actually, each composition in (55) is null (since, up to d-homotopy equivalence, every map in the fibre sequence (51) is the h-kernel of the following one). On the other hand, take $[x] \in \uparrow\pi_0 X$ such that $f_{*0}[x] = [f(x)] = [*_Y]$; this only means that there is a finite sequence of paths in Y

$$f(x) = y_0 \to y_1 \leftarrow y_2 \to \ \dots\ y_n = *_Y, \tag{56}$$

but does not give, generally, a path $*_Y \to f(x')$ with $[x'] = [x]$.

If f is a bilateral fibration (1.7) we do get exactness at $\uparrow\pi_0 X$ (but nothing more). In fact, we can lift in X all the previous paths

$$x = x_0 \to x_1 \leftarrow x_2 \to \ \dots\ x_n, \qquad f(x_n) = *_Y, \tag{57}$$

so that $p_{*0}\colon \uparrow\pi_0(\uparrow K^- f) \to \uparrow\pi_0(X)$ takes $[x_n, 0_{*_Y}]$ to $[x_n] = [x]$.

4. Based directed spaces

Generalising pointed objects, one can keep fixed a *set* of base-points, with the same advantage of controlling homotopy invariance *at* them.

4.1. Bipointed objects. Let us begin considering the 'comma' category $\mathrm{d}\mathbf{Top}\backslash\mathbf{S}^0 = (\mathbf{S}^0{\downarrow}\mathrm{d}\mathbf{Top})$ of *bipointed* d-spaces. An object (X, x, x') is a d-space equipped with a pair of points (possibly equal); bipointed maps preserve them and bipointed (directed) homotopies leave each of them fixed.

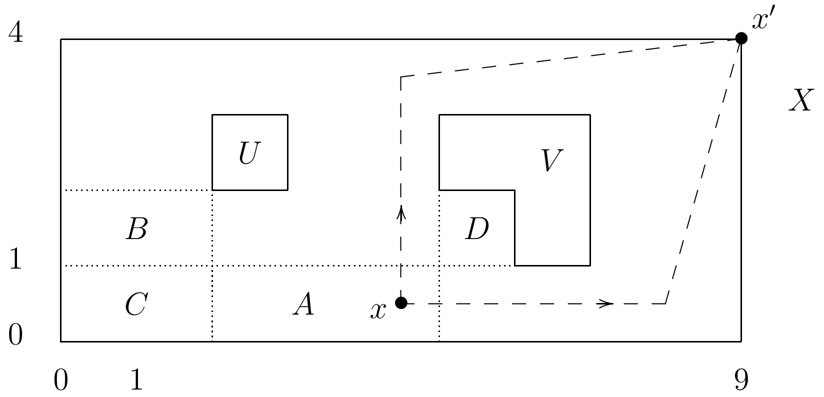
We already know (from I.3.3.2) that the homotopy functor

$$\begin{aligned} &\uparrow\pi_1\colon \mathrm{d}\mathbf{Top}\backslash\mathbf{S}^0 \to \mathbf{Set}, \\ &\uparrow\pi_1(X, x, x') = \uparrow\Pi_1(X)(x, x') = [(\uparrow\mathbf{I}, 0, 1), (X, x, x')]_{\mathbf{S}^0}, \end{aligned} \tag{58}$$

is (strictly) invariant up to bipointed d-homotopy; this is also evident by its presentation as a representable homotopy functor, in the last formula above.

For instance, one can examine by such functors the following d-space $X \subset \uparrow\mathbf{R}^2$

$$X = ([0,9]\times[0,4]) \setminus (U \cup V)$$



(59)

Keeping x' fixed at $(9,4)$ and letting x vary in X, we get different results for the set $\uparrow\pi_1(X, x, x')$: two equivalence classes if x is in the rectangle A or in B, three if it is in C, none in D, one in the rest of X. As it happens for the fundamental monoid (3.4), *each such information is invariant up to bipointed d-homotopy and can be computed via the fundamental category.*

The ordered circle $\uparrow\mathbf{O}^1$, bipointed at minimum and maximum, yields a representable homotopy functor $[\uparrow\mathbf{O}^1, -]\colon \mathrm{d}\mathbf{Top}\backslash\mathbf{S}^0 \to \mathbf{Set}$, which counts the 'linked pairs of paths' in X, from x to x', up to bipointed homotopy. But the obvious symmetry $\uparrow\mathbf{O}^1 \to \uparrow\mathbf{O}^1$ produces a symmetry on $[\uparrow\mathbf{O}^1, X]$, i.e. an action of the group $\mathbf{Z}_2$, and we may prefer to count those pairs of paths up to this action

$$O_1\colon \mathrm{d}\mathbf{Top}\backslash\mathbf{S}^0 \to \mathbf{Set}, \qquad O_1(X, x, x') = ([(\uparrow\mathbf{O}^1, v^-, v^+), (X, x, x')]_{\mathbf{S}_0})/\mathbf{Z}_2; \qquad (60)$$

it might be interesting to develop methods of deducing this invariant from $\uparrow\Pi_1 X$.

4.2. Based objects. More generally, let A be a fixed d-space, called the basis, and consider the comma category $\mathrm{d}\mathbf{Top}\backslash A = (A{\downarrow}\mathrm{d}\mathbf{Top})$ of *d-spaces under* A. An object is a pair (X, x), where $x\colon A \to X$ is a d-map; a map $f\colon (X, x) \to (Y, y)$ is defined by a d-map $f\colon X \to Y$ such that $f.x = y$. A *based* (directed) homotopy $\varphi\colon f \to g\colon (X, x) \to (Y, y)$ is a homotopy $\varphi\colon f \to g$ in $\mathrm{d}\mathbf{Top}$ which is 'trivial on the basis', i.e. $\varphi\circ x = 0_y$ (each path $\varphi(x(i), -)\colon \uparrow\mathbf{I} \to Y$ is constant, for $i \in A$).

The category $\mathrm{d}\mathbf{Top}\backslash A$ acquires thus a d-homotopy structure similar to that of $\mathrm{d}\mathbf{Top}$, by suitable cylinder and cocylinder (as proved - in a more complex situation - in [8], Section 4; the crucial point is the fact that pushouts in $\mathrm{d}\mathbf{Top}$ exist and are preserved by $\uparrow I$, a left adjoint).

We shall only recall how to construct the based cylinder and cocylinder. The latter is obvious

$$\uparrow P(X,x) = (\uparrow PX, x^P), \qquad x^P = \uparrow Px.e\colon A \to \uparrow PA \to \uparrow PX, \tag{61}$$

while the cylinder $\uparrow I(X,x)$ is obtained from a pushout in d**Top**, which collapses (independently) each fibre at a point $x(i)$, for all $i \in A$

$$\begin{array}{ccccc} A & \overset{\partial^\alpha}{\dashrightarrow} & \uparrow IA & \xrightarrow{\;e\;} & A \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle \uparrow Ix} & & \downarrow{\scriptstyle x^I} \\ X & \underset{\partial^\alpha}{\dashrightarrow} & \uparrow IX & \longrightarrow & \uparrow I(X,x) \end{array} \tag{62}$$

(to be precise, $\uparrow I(X,x)$ is in fact this pushout object *equipped* with the d-map x^I).

The fundamental category can then be viewed as a functor

$$\uparrow\Pi_1\colon \mathrm{d}\mathbf{Top}\backslash A \to \mathbf{Cat}\backslash {\uparrow}\Pi_1(A). \tag{63}$$

In particular, if A is a set (viewed as a discrete d-space and a discrete category), we have

$$\uparrow\Pi_1\colon \mathrm{d}\mathbf{Top}\backslash A \to \mathbf{Cat}\backslash A, \qquad \uparrow\Pi_1(X,x) = (\uparrow\Pi_1 X, x_*), \tag{64}$$

a functor invariant up to A-based d-homotopies, *of d-spaces and of categories* (trivial on each point of A, in both domains). Again, the crucial fact is that *this functor cannot destroy information at the base points* $x(i)$, for $i \in A$, (in contrast with the free $\uparrow\Pi_1$ on d**Top**, cf. I.3.5).

This follows immediately from the invariance theorem I.3.2. First, if $\varphi\colon f \to g$ is A-based, then each path $\varphi(x(i))\colon f(x(i)) \to g(x(i))$ is trivial. Second, if $f\colon (X,x) \to (Y,y)$ is an A-based d-homotopy equivalence, then

$$f_{*1}\colon \uparrow\Pi_1(X)(x(i),x(j)) \to \uparrow\Pi_1(Y)(fx(i),fx(j)) \tag{65}$$

is bijective (for $i,j \in A$).

4.3. Variable basis. Finally, we want to be able to change bases, and to have morphisms between objects with different bases. This is solved by the category of morphisms d$\mathbf{Top}^{\mathbf{2}}$, already considered in 2.1.

An object is a d-map $x\colon X' \to X''$, and a morphism is a commutative square in d**Top**

$$f = (f',f'')\colon x \to y, \qquad f'\colon X' \to Y', \qquad f''\colon X'' \to Y''; \qquad yf' = f''x. \tag{66}$$

Again, the homotopy structure of this category of functors $\mathbf{2} \to \mathrm{d}\mathbf{Top}$ has already been studied, in general, in [8], Section 4. It is of the same type as the one of d**Top**, with the obvious derived cylinder and cocylinder functors, whose values on objects are, respectively

$$\begin{array}{l} \uparrow I(x\colon X' \to X'') = (\uparrow Ix\colon \uparrow IX' \to \uparrow IX''), \\ \uparrow P(x\colon X' \to X'') = (\uparrow Px\colon \uparrow PX' \to \uparrow PX''). \end{array} \tag{67}$$

The fundamental category can then be viewed as a functor

$$\uparrow\Pi_1\colon \mathrm{d}\mathbf{Top^2} \to \mathbf{Cat^2}. \tag{68}$$

Also here, it is interesting to restrict to the case where the domain of our d-maps $x, y, \ldots$ is a (variable) *set*, viewed as a discrete d-space via the functor $D\colon \mathbf{Set} \to \mathrm{d}\mathbf{Top}$ (or a discrete category, via $D\colon \mathbf{Set} \to \mathbf{Cat}$). We obtain then the comma category $(D{\downarrow}\mathrm{d}\mathbf{Top}) \subset \mathrm{d}\mathbf{Top^2}$: an object is a d-map $x\colon DA \to X$, a map (f_0, f) consists of a set-mapping f_0 and a d-map f forming a commutative square (Df_0 is written as f_0)

$$\begin{array}{ccc} DA & \xrightarrow{f_0} & DB \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad (f_0, f)\colon (x\colon DA \to X) \to (y\colon DB \to Y). \tag{69}$$

A *based d-homotopy* $\varphi\colon (f_0, f) \to (g_0, g)\colon x \to y$ can only exist if $f_0 = g_0$, and consists then of a d-homotopy $\varphi\colon f \to g$ such that $\varphi \circ x = 0_{yf_0}$. The category $(D{\downarrow}\mathbf{Cat})$ is similar.

The fundamental-category functor between such comma categories

$$\uparrow\Pi_1\colon (D{\downarrow}\mathrm{d}\mathbf{Top}) \to (D{\downarrow}\mathbf{Cat}), \qquad \uparrow\Pi_1(x\colon DA \to X) = (x\colon DA \to \uparrow\Pi_1 X). \tag{70}$$

associates to the above homotopy φ of $(D{\downarrow}\mathrm{d}\mathbf{Top})$, a natural transformation $\varphi_{*1}\colon f_{*1} \to g_{*1}$ which is the identity on all objects $x(i)$ (for $i \in A$): $\varphi_{*1}x(i) = \mathrm{id}\colon y(f_0 i) = y(g_0 i)$. Therefore

$$f_{*1} = g_{*1}\colon \uparrow\Pi_1(X)(x(i), x(j)) \to \uparrow\Pi_1(Y)(y(f_0 i), y(f_0 j)) \qquad (i, j \in A), \tag{71}$$

and, again, all sets $\uparrow\Pi_1(X)(x(i), x(j))$ are (bijectively) invariant up to based d-homotopy equivalence.

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Dipartimento di Matematica
Università di Genova
Via Dodecaneso 35
16146-Genova, Italy
Email: grandis@dima.unige.it

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